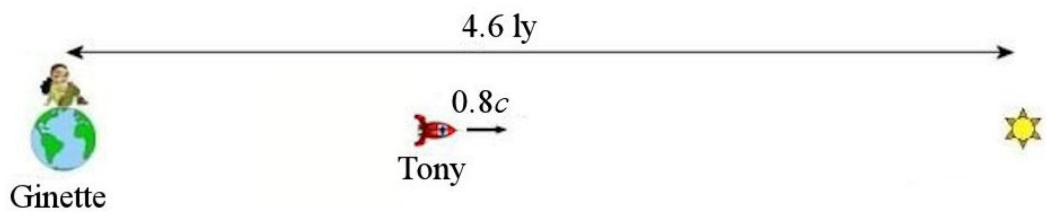
9 RELATIVITY

Ginette stays on Earth while Tony travels towards a star located 4.6 light-years away from Earth. The speed of Tony's ship is 80% of the speed of light.



a) What is the duration of the trip according to Ginette?

b) What is the duration of the trip according to Tony?

hdwpapers.com/spaceship_wallpaper_hd-wallpapers.html

Discover how to solve this problem in this chapter.

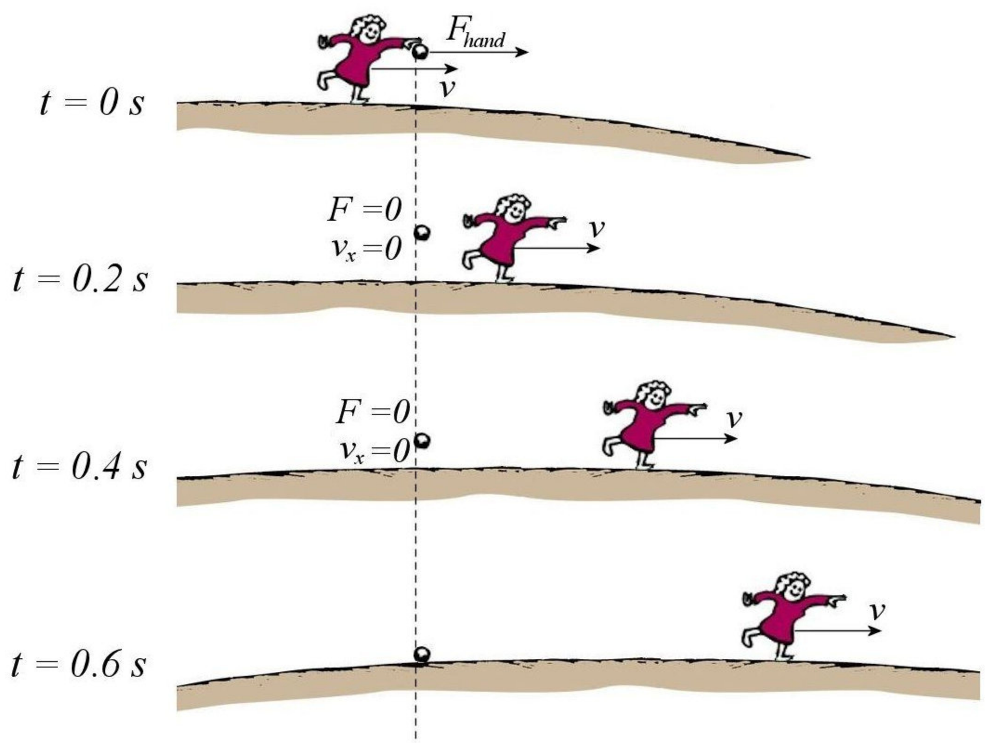
9.1 THE RELATIVITY PRINCIPLE

The Debate on the Motion of the Earth

Significant Effects According to Prenewtonian Physics

Before the 17th century, very few people could accept the idea of the Earth moving around the Sun. Many opposed this theory because of the effects we were supposed to observe according to the physics of the time if the Earth is revolving around the Sun.

In prenewtonian physics, speed is associated with force. This means that a force must be applied on an object in order to move it. To illustrate the consequences of such a physics, let's imagine that someone drops a stone on the ground while the Earth is moving. As long as the stone is touching the hand of the person, it could be argued that the hand of the person is exerting a force on the stone that allows it to move at the same speed as the Earth. However, when the person drops the stone, there would be no longer any horizontal force acting on the stone. According to prenewtonian physics, this means that the stone would no longer be able to move forwards. The stone would, therefore, lose its horizontal velocity while the Earth would continue its motion. If it is assumed that the duration of the fall of the stone is 0.6 seconds, then we would have the following situation.



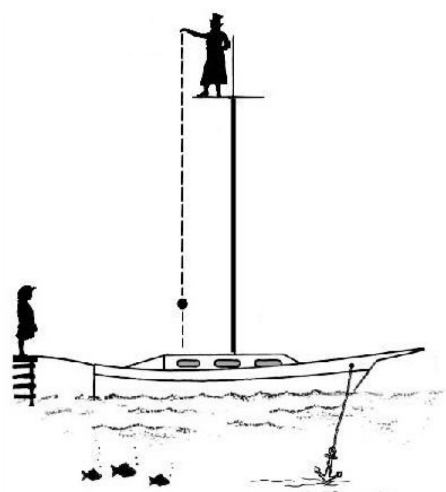
slid.es/tofergregg/gravity-and-fluid-dynamics/fullscreen#/22

Thus, the person on Earth sees the stone fall behind him. Actually, the stone would fall very far behind since the Earth is moving at nearly 30 km/s. If it took 0.6 seconds for the stone to fall, then the Earth would have moved 18 km while the stone remained at the same *x*-coordinate. Therefore, the stone would fall 18 km behind the person. As objects do not fall far behind us when we let them fall, this showed that the Earth is not moving (according to the physics of the time which associates force and velocity).

With similar arguments, a series of effects that should be observed if the Earth is moving can be predicted. For example, there should be a continuous wind on the surface of the Earth. The air surrounding the Earth cannot move with the Earth because it is hard to imagine how a force could push the air to move it. So, the Earth is supposed to be moving in air at rest, which would give the impression that there is a continuous wind in the opposite direction of the motion of the Earth. It is certain that the presence of such a continuous 30 km/s wind would be noticed.

Galileo's Arguments in Favour of the Motion of the Earth

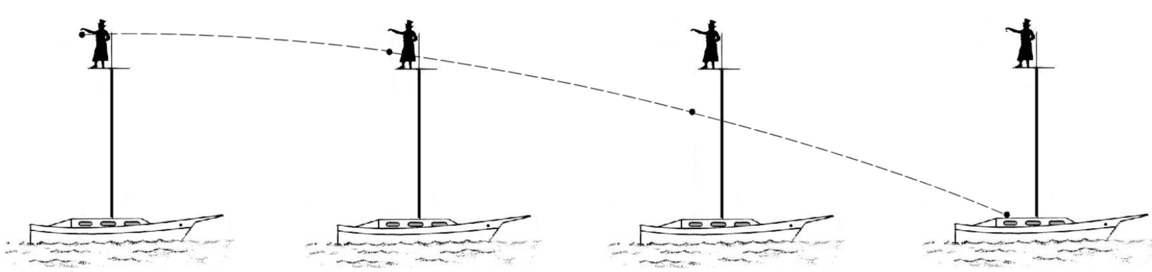
Galileo was the first to find convincing arguments to show that the Earth can be moving without generating any dramatic effects. The exact reasoning of Galileo will not be followed here because he used prenewtonian physics with impregnated forces, but the conclusions will be the same.



www.relativityoflight.com/Chapter5.html

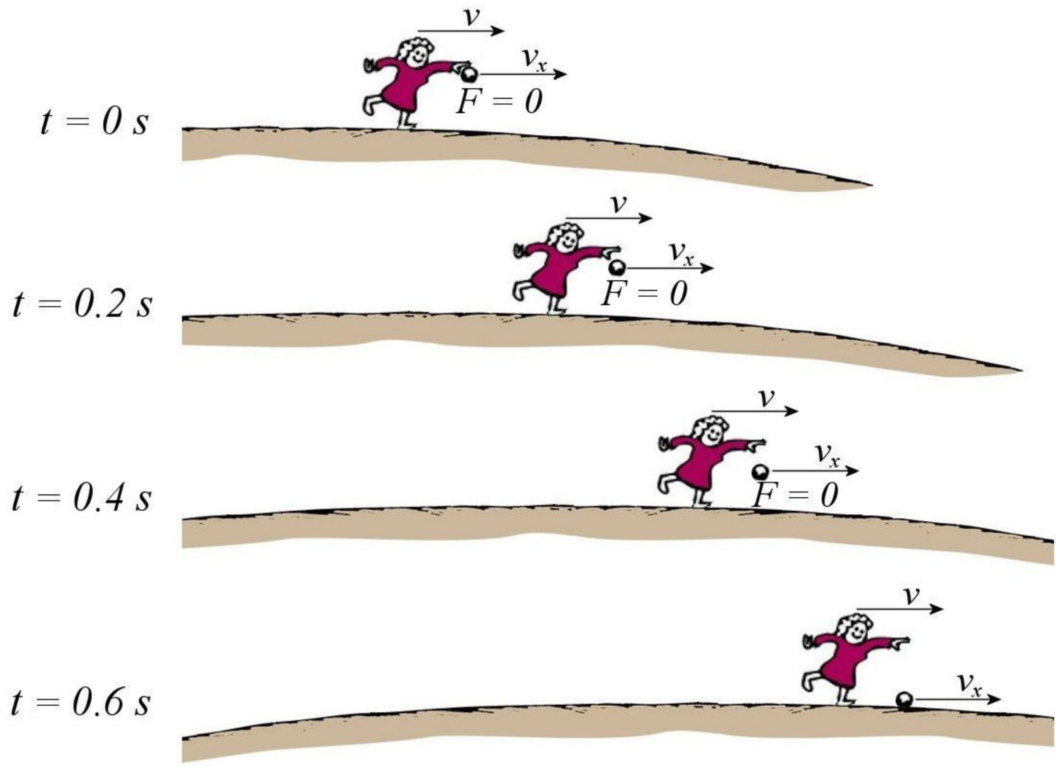
Imagine what happens when Valteri drops a stone from the top of the mast of a ship when the ship is at rest. In this case, the stone strikes the cabin just above the last window.

Now, let's look at what happens if Valteri drops a stone from the top of the mast when the ship is moving. Some, intuitively using prenewtonian physics, would say that the stone will fall a little towards the back of the ship because the ship is moving forward while the stone is falling. This isn't going to happen. When Valteri let go of the stone, the stone already has a horizontal velocity equal to the velocity of the boat. As no horizontal force acts on the stone, it always keeps this same horizontal velocity while falling. The stone is thus moving forward at the same speed as the boat and the resulting motion is shown in the next diagram (where the boat is moving quite fast).



First of all, it can be noted that the object hits the boat at exactly the same place as it did when the boat was at rest: above the last window. Viewed from the boat, the fall is in fact identical to what was seen when the boat was at rest: as the stone falls, it always stays at the same distance from the mast and hits the cabin above the last window.

Using such arguments, it is possible to argue that it is possible to be on a planet moving at a constant speed without seeing any effects. Let's use the example of the person dropping a stone on the surface of the Earth to show this.



When the stone is in the hand of the person, the hand exerts no horizontal force on the stone. As the stone is going at the same speed as the Earth and as this speed is constant, there is no acceleration and, therefore, no force.

When the stone falls, there is still no horizontal force acting on it. This means that there is no horizontal acceleration, and that the horizontal velocity of the stone remains the same. The x-component of the velocity of the stone is always identical to the velocity of the Earth, and the stone advances at the same rate as the Earth. This means that the stone will hit the ground in front of the person, exactly under the hand of the person. The same thing would happen is the Earth was not moving.

In fact, it can be shown that everything that happens on a planet moving at a constant speed is identical to what happens on a planet at rest. With such arguments, the idea of a moving Earth became a bit more popular (but the majority of the scientists of the time still did not accept that the Earth is moving before the discovery of Newton's laws).

The Relativity Principle

These arguments are the basis of the principle of relativity. It was just shown that a person on the ship sees the stone fall exactly in the same way whether the boat is at rest or moving at a constant speed. This means that there are no new effects that appear when moving at a constant speed. Everything happens as if the person were at rest.

For example, we know how to pour a cup of coffee in our house (which is at rest). If, one day, you become a flight attendant and you must pour coffee into a cup in a plane moving at a constant speed, do not change your habits: pour the coffee on the plane in motion exactly the same way as you do it at home. There is no need to pour the coffee a bit in front of the cup, thinking that the coffee will move a little towards the rear of the aircraft because of its motion. Pool (the game) is played exactly the same way on a plane travelling at a constant speed as in a pool hall. There is no "correction" to do because of the motion of the plane.

In this video, a ball is dropped from a moving truck. A board was installed on the side of the truck to give a reference for the motion of the ball. The ball stays in the middle of the board as it falls, exactly as what would happen if the ball were released while the vehicle was stopped.
http://www.youtube.com/watch?v=_ky-ITbNfeY

All this also means that if an observer is locked inside a plane in motion at a constant speed without any window, it is impossible for the observer to tell whether the plane is moving or not. If no new effect appears when in motion, everything appears exactly the same way whether the aircraft is moving or not. No experiment can show that it is moving.
http://www.youtube.com/watch?v=uJ8l4kh_jto

This also means that if you close your eyes in a car in motion (if you're not driving of course), you won't be able to say whether the car is moving or not. You surely think that any moron can tell whether the car is moving or is stopped, and that this is not true. Actually, it is true. You would know that the car moves because the bumps on the road will shake the car a little. However, we're saying here that the motion is undetectable if the speed is constant. If that is the case, you must imagine that you are travelling on a road without any bumps because these bumps accelerate the car in every direction. So you must imagine that you are travelling on a freshly paved road and that there is no bump whatsoever. Then, it becomes more difficult to tell if the car is moving. You still hear the air passing along the car, but this could be a wind blowing on the car at rest… It would be more convincing with a spaceship. Then, there would be no sound from the wind or the road and it would be impossible to tell whether the spaceship is moving or not if your eyes are closed.

The final argument that should convince you that nothing changes when moving at a constant speed is the fact that we are a planet moving around the Sun at 29.8 km/s and nothing peculiar happens. Even better, the Sun revolves around the centre of the Galaxy at

240 km/s, and we do not notice this motion. Everything happens as if the Earth were at rest.

If there are no new effects appearing at a constant speed, it is because the laws of physics are the same for a moving observer. There are no new forces that come into play, which means that the laws are exactly the same. This brings us to the Galilean principle of relativity (actually stated by Euler).

The Relativity Principle

The laws of physics are the same for all the observers moving at a constant velocity.

9.2 GALILEAN TRANSFORMATIONS

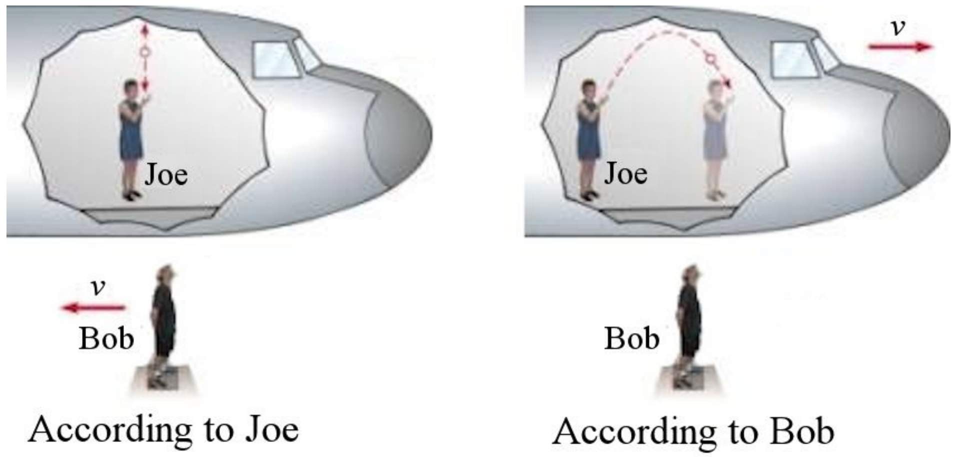
Description of a Phenomenon by two Observers

It has been said that this principle means that we cannot tell if our spacecraft is moving or not if we close our eyes when the spacecraft is moving at a constant velocity. But if we open our eyes, we will see that the spaceship is moving when we see the planets moving around us. We will see the planets move, but that doesn't necessarily mean that the ship is moving. You could very well say that the spacecraft is at rest and that the planets are moving. According to the principle of relativity, this is a perfectly valid explanation. The point of view in which the ship is not moving and the planets are moving is just as good as the point of view in which the ship is moving and the planets are not moving.

With the principle of relativity, you can therefore say that your car is stopped, and that the road and the entire landscape are moving when you go for a ride. This is a perfectly acceptable point of view according to the principle of relativity. This is just as valid a view as the one where the Earth is stopped, and your car is moving.

Since all points of view are equality good in relativity, a phenomenon can be analyzed from any point of view using the same laws of physics, and the result obtained will always be valid. There is no point of view that is better than another. Analyzing the movement of a car from the point of view where the car is stopped and the ground is moving at 100 km/h is just as good as taking the point of view where the ground is at rest and the car is moving at 100 km/h (which is what we usually do). We could even take a point of view where the Earth is going at 50 km/h in one direction and the car at 50 km/h in the other direction.

For example, suppose Joe, on a plane, throws a ball into the air and catches it. Bob is on the ground and observes the motion of the ball thrown by Joe. Both observers will describe this motion from their point of view.



www.physics.uc.edu/~sitko/CollegePhysicsIII/26-Relativity/Relativity.htm

For Joe, the ball moves in a straight line, upwards and then downwards. If Bob, who is on the ground, look at what Joe is doing, he will say that the ball made a parabolic motion. The descriptions are different, but the law for this phenomenon is the same for two observers: the ball falls down with a 9.8 m/s² downwards gravitational acceleration.

Is one description better than the other? Not at all according to the principle of relativity. The description of the motion of the ball according to Joe is different from the description of the motion of the ball according to Bob, but it is just as good. All observers are on the same footing according to relativity. Joe will say that the *x*-component of the velocity of the ball is zero, and Bob will say that this component is not zero. Who is right? Both are right. The initial conditions of the motion can be different from one observer to another, but the laws of physics are the same. This means that even if Bob and Joe do not agree with each other on the initial velocity, they agree on the fact that the acceleration of the ball is 9.8 m/s² downwards.

Some Definitions

An **event** is something that occurs at a specific time and place. It could be an explosion, for example. To describe an event, the position and the time at which the event occurred must be given. If I tell you about an event that occurred on April 15, 1912, at 2:20 at position 41°43′57″ N 49°56′49″ O in the Atlantic Ocean, you know (maybe) that I'm talking about the sinking of the Titanic. So, an event is noted by its position and time coordinates (x, y, z, t). The event doesn't have to be spectacular. Someone sneezing is an event, and the position and time of this event can be noted.

An **observer** is someone who notes the position and the time at which events occur. Observers can be moving relative to each other at a constant velocity. According to the principle of relativity, no observer is better than another. All observers have different views, but they all are equally valid.

Each observer uses an axes system which is different from the system used by other observers. These are the **reference frames**. If an observer is moving at a certain speed, its axes system moves at the same speed as the observer. Although we don't have to necessarily do it, we can consider that each observer is always at the origin of his axes system. Each observer will, therefore, note the position and the time of the events using his frame of reference. Each observer notes different coordinates, since the origins are not at the same place for each observer.

There is a difference here between **seeing** an event and **observing** it. To illustrate this difference, imagine that a 5 light-years distant star exploded in 2012. (The light-year is a unit of distance which corresponds to the distance travelled by light in one year. Since the light travels at 300 000 km/s, this distance is 9.46×10^{15} m). Then, it takes 5 years for the light from the explosion to reach us, and the explosion is seen in 2017. However, when we see the explosion in 2017, we can deduce that the explosion actually occurred in 2012. So, we **see** the explosion in 2017 but we **observe**, with a little math, that it has occurred in 2012.

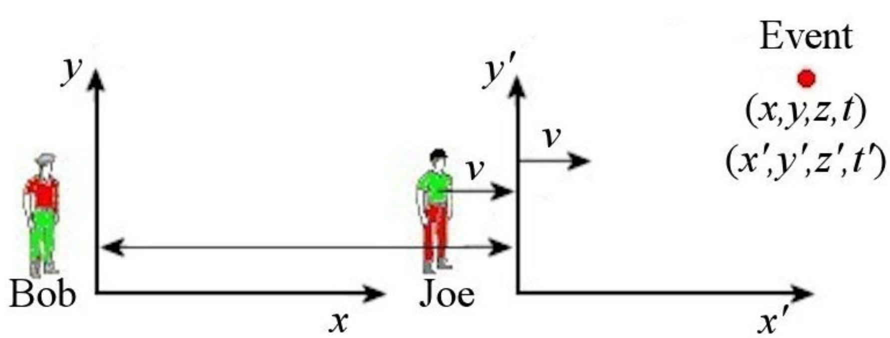
(Actually, the calculation can be a little more complicated than that if the source is moving. Then, we must find the position of the source when the light coming from it left. The distance of the source when the light was emitted must be used, not the current distance of the source.)

Transformation Laws

We now want to shift from one observer to another. For example, we want to know the position of an event according to one observer if the position of an event according to another observer is known.

Positions Transformations

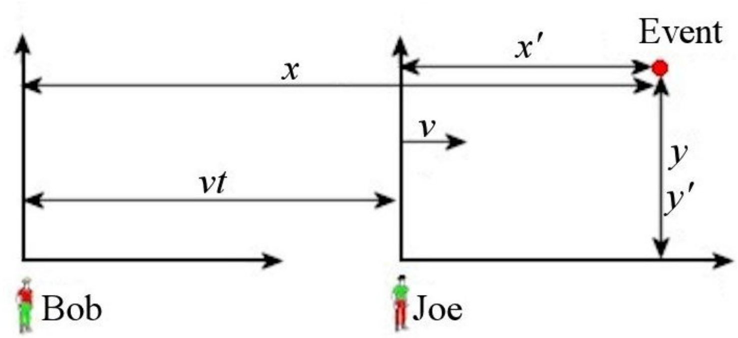
Two observers observe an event. Bob is at rest, and Joe moves at a constant speed v towards the right. Each of these observers notes the position and the time of the event with different axes systems.



en.wikibooks.org/wiki/Special_Relativity/Print_version

Bob's axes system is stationary while Joe's axes system follows Joe and, therefore, travels at a constant speed *v* towards the right. Obviously, the two observers will not note the same values for the position of the event since the origins of the axes systems they use are not at the same place. They will, therefore, disagree on the coordinates of the position of the event.

Fortunately, laws to pass from Bob's coordinates to Joe's coordinates (or vis-versa) can be obtained. If Joe tells Bob the coordinates of the event according to him, then Bob can calculate the coordinates according to his own axes with these formulas.



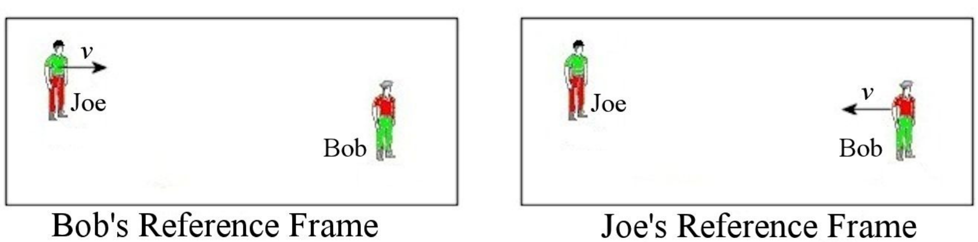
The distance between the two *y*-axes always increases since Joe moves towards the right. Assuming that the two *y*-axes were at the same place at $t = 0$, the distance between the axes is vt. The distance between the *y*-axis and the event is *x,* and the distance between the *y′*-axis and the event is *x′*. Similarly, the distance between the *x*-axes of both observer and the event gives the values of *y* (*y* and *y′*). From the diagram, it is easy to find that

$$x = x' + vt$$
$$y = y'$$

These are the transformation laws. They are called *Galilean transformations*. They can also be inverted to calculate *x′* and *y′* from *x* and *y*.

$$x = x' + vt \qquad x' = x - vt$$
$$y = y' \qquad y' = y$$
$$z = z' \qquad z' = z$$

There is a convention to determine which of the two observers notes the coordinates with primes. In our situation, Bob sees Joe travelling towards the positive *x*-axis. If the point of view of Joe is taken, Bob is travelling towards the negative *x*-axis.



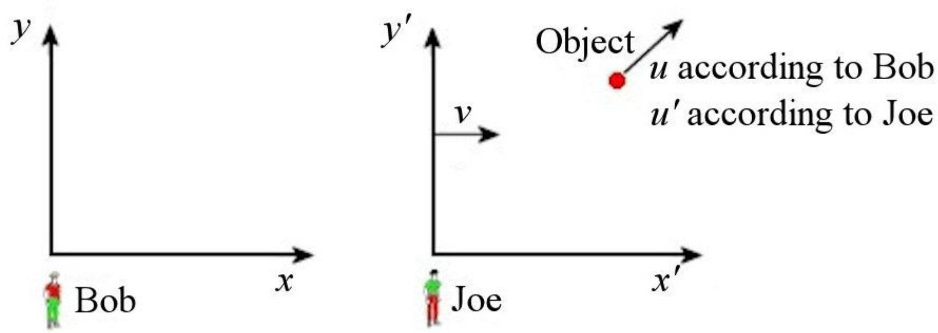
The rule is

The observer who sees the other travelling towards the
negative x-axis uses primes

So, Joe uses primes here. (Assuming, of course, that the x-axis is pointing towards the right.)

Velocity Transformations

Suppose 2 observers observe a moving object. We want to know how to obtain the speed of the object according to one observer from the speed measured by the other observer. The speed of the object is denoted u (because v is already used to indicate the speed of Joe with respect to Bob).



The velocity of an object corresponds to the rate at which its position changes. Therefore, according to Bob, the components of the velocity are

$$u_x = \frac{dx}{dt} \qquad u_y = \frac{dy}{dt}$$

According to Joe, the components are

$$u'_x = \frac{dx'}{dt} \qquad u'_y = \frac{dy'}{dt}$$

If the Galilean transformation $x = x' + vt$ is used, the result is

$$\frac{dx}{dt} = \frac{d\left(x' + vt\right)}{dt}$$
$$\frac{dx}{dt} = \frac{dx'}{dt} + \frac{d\left(vt\right)}{dt}$$
$$u_x = u'_x + v$$

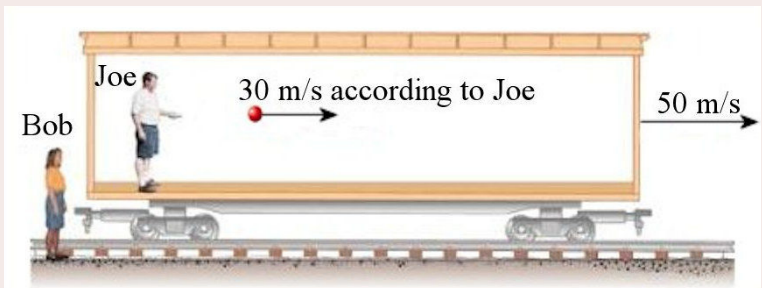
Doing the same for the Galilean transformation for the coordinates y and z, the following Galilean laws of transformation for the velocities are obtained.

$$u_x = u'_x + v \qquad u'_x = u_x - v$$
$$u_y = u'_y \qquad u'_y = u_y$$
$$u_z = u'_z \qquad u'_z = u_z$$

Once again, the two observers disagree on the velocity of objects. However, this is logical as the next example illustrates.

Example 9.2.1

Joe, who is on a train, throws a ball towards the front of the train. If the speed of the ball according to Joe is 30 m/s and if the train is going at 50 m/s according to Bob, what is the speed of the ball according to Bob?



www.physics.uc.edu/~sitko/CollegePhysicsIII/26-Relativity/Relativity.htm

The speed of the ball according to Joe is $u'_x = 30$ m/s.

(The speed of the object according to the observers is always denoted with the letter u. As Joe works with the axes x' and y', everything that Joe measures has a prime.)

The speed between the two observers is $v = 50$ m/s.

(v is always the speed between the two observers.)

We are looking for the speed of the ball according to Bob which is u_x.

(The speed of the object according to observers is always u. As Bob works with the axes x and y, everything that Bob measure doesn't have a prime.)

According to the laws of transformation of velocities, the speed is

$$\begin{aligned} u_x &= u'_x + v \\ &= 30 \tfrac{m}{s} + 50 \tfrac{m}{s} \\ &= 80 \tfrac{m}{s} \end{aligned}$$

This makes sense. A ball thrown at 30 m/s towards the front of a train moving at 50 m/s has a total speed of 80 m/s for an observer on the ground who is looking at the train.

This video illustrates these velocity transformations.
http://www.youtube.com/watch?v=iHfCwJwdIv8

It is no big deal if the two observers do not measure the same speed since Newton's laws are not centred on speeds. Even if there are some quantities that depend on speed in

Newton's physics, the principles of Newtonian physics are not affected. For example, two observers will disagree on the momentum of the objects because the speeds are different, but this does not matter since they will still agree on the conservation of the total momentum of a system, and this is what is important in Newton's physics.

Acceleration Transformations

For Newton's laws to be respected, the observers must agree on the acceleration of an object. Since the forces must be identical for the two observers, then the accelerations must also be the same for both observers. A difference would mean that Newton's laws are valid only for one of these two observers and that there would be restrictions on the use of Newton's laws. Let's check this by finding the acceleration transformation laws. The acceleration of the object, which is the rate at which the velocity changes, is, according to Bob,

$$a_x = \frac{du_x}{dt} \qquad a_y = \frac{du_y}{dt}$$

According to Joe, the acceleration is

$$a'_x = \frac{du'_x}{dt} \qquad a'_y = \frac{du'_y}{dt}$$

The transformation equation for the accelerations is obtained by deriving the transformation equation for the velocity. For the x-component, the result is

$$\frac{du_x}{dt} = \frac{d\left(u'_x + v\right)}{dt}$$
$$\frac{du_x}{dt} = \frac{du'_x}{dt} + \frac{d\left(v\right)}{dt}$$
$$a_x = a'_x$$

The derivative *dv/dt* is zero since Joe's speed is constant. Proceeding in the same manner for the transformation of the other components of the velocity, the following Galilean transformation laws for the acceleration are obtained.

$$a_x = a'_x \qquad a'_x = a_x$$
$$a_y = a'_y \qquad a'_y = a_y$$
$$a_z = a'_z \qquad a'_z = a_z$$

The components of the accelerations are the same for both observers. This means that the forces, which are, of course, identical according to both observers, give the same acceleration. This implies that Newton's laws are valid for both observers: the one at rest and the one moving at a constant speed. This means that Newtonian mechanics is consistent with the principle of relativity. A person who moves at a constant speed can apply Newton's laws exactly the same way as a person at rest.

The Same Experiment According to 2 Observers

The principle of relativity specifies that the same experiment done by 2 observers with different speeds must give exactly the same results. For example, if Joe in an airplane that is moving relative to the ground launches a projectile with a certain initial velocity, the trajectory must be identical to the trajectory obtained by Bob on the ground who launches a projectile with the same initial velocity. (Be careful, here the two observers are not looking at the same experiment. Each observer does an experiment by each throwing a ball with the same initial velocity as measured by the person throwing the ball.)

According to Bob, the position of his ball on its parabolic trajectory is denoted (x,y). According to Joe, the position of his ball on its parabolic trajectory is denoted (x',y'). According to the principle of relativity, the 2 trajectories must be identical, which means that the (x,y) positions measured by Bob must be identical to the (x',y') positions measured by Joe.

Here's what Bob will predict. The launch of Joe's ball is identical to the launch of Bob's ball, except that its initial velocity in x is greater by v according to Bob (since the ball already has the velocity of the airplane at launch according to Bob). Bob can therefore find the trajectory of Joe's ball according to Joe by finding the trajectory of his own ball according to Joe and then adding the velocity v to the x-component of the velocity of the ball.

The position of Bob's ball according to Joe is given by the Galilean transformations.

$$x' = x - vt$$
$$y' = y$$

If Bob now adds the extra initial velocity v to the x-component, he gets

$$x' = x - vt + vt$$
$$y' = y$$

The vt cancel each other out, and the positions are the same! The trajectories are, therefore, identical for the 2 observers as specified by the principle of relativity.

Using the Relativity Principle

The solution of certain problems can be simplified, and some interesting results can be obtained from the relativity principle. Here are two examples to illustrate.

For example, collision problems can be solved more easily by taking the point of view of another observer.
http://physique.merici.ca/waves/collisionrel-eng.pdf

Interesting demonstrations can also be obtained. For example, here is the demonstration that the momentum must be conserved if the kinetic energy is conserved in a collision.
http://physique.merici.ca/waves/ConservationEp1-eng.pdf

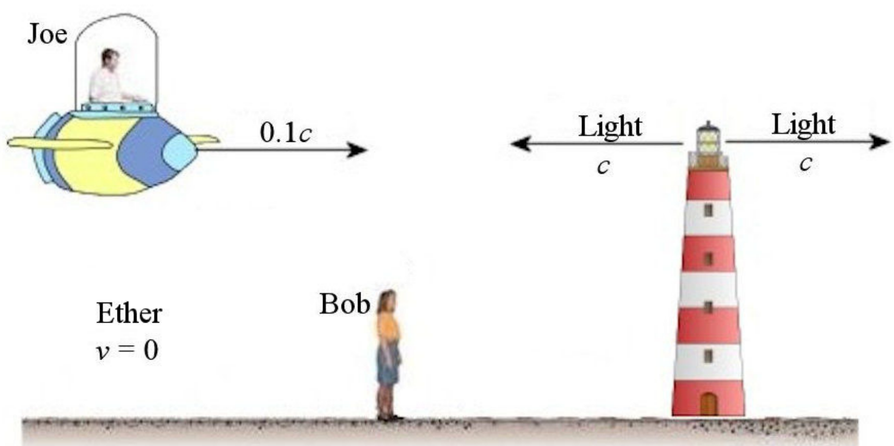
9.3 PROBLEMS WITH LIGHT

The Ether and Speed of Light

At the end of the 19th century, everyone was convinced that light was a wave. Moreover, according to the models of the time, light was supposed to be propagating in a medium called *ether* which was present everywhere in the universe. The light therefore propagated at 300 000 km/s relative to the ether. However, there was a serious problem with the movement of Earth in ether[1].

To understand the problem, let's look at what Galileo's relativity predicts when two observers observe a beam of light propagating in ether. One observer, Bob, is at rest in ether while the other observer, Joe, travels at $0.1c$ in ether.

In Bob's frame, we have



www.physics.uc.edu/~sitko/CollegePhysicsIII/26-Relativity/Relativity.htm
www.tuxpaint.org/stamps/index.php3?cat=town&page=3
www.jrj-socrates.com/Cartoon%20Pages/kaput_and_zosky.htm

[1] Kelvin is often quoted as saying in 1900 that everything was discovered and that there were only two little clouds left in the world of physics. In fact, Kelvin did not say as much as this. In 1900 Kelvin made no reference to physics in general, but only to mechanistic models (i.e. attempts to explain the world by mechanisms like ether) in optics and thermodynamics. He then pointed to the problems of Earth's motion in ether and to the difficulty the concept of equipartition of energy theorem posed for the construction of molecular models.

According to Bob, the speed of light is 300 000 km/s in every direction.

Now, let's look at the same situation by taking the point of view of Joe and calculate the speed of both light beams according to Joe.

Beam travelling towards the right.

Speed of the beam according to Bob: $u_x = c$
Speed of the beam according to Joe: $u_x' = ?$
Relative speed of Joe and Bob: $v = 0.1c$

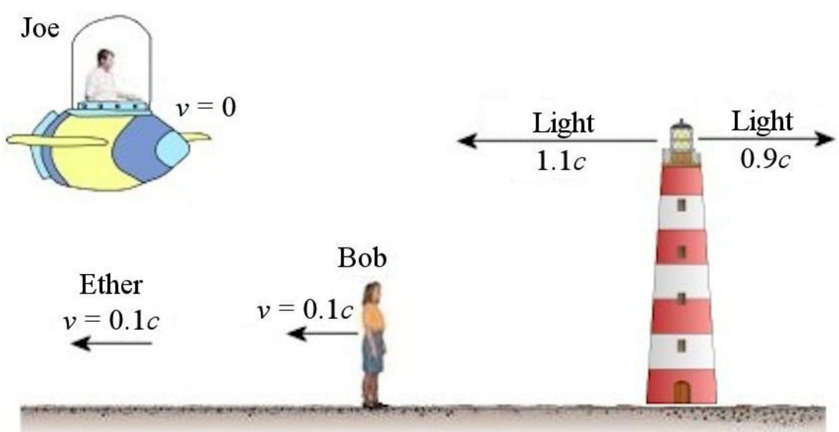
$$u_x' = u_x - v = c - 0.1c = 0.9c$$

Beam travelling towards the left.

Speed of the beam according to Bob: $u_x = -c$
Speed of the beam according to Joe: $u_x' = ?$
Relative speed of Joe and Bob: $v = 0.1c$

$$u_x' = u_x - v = -c - 0.1c = -1.1c$$

So, this is the situation as seen from Joe's frame of reference.



This result is not really surprising. According to Joe, the light going towards the right is slowed by the motion of the ether towards the left, somewhat like a fish swimming against the current that goes slower than when there is no current. The light going towards the left is driven by the motion of the ether towards the left, giving it greater speed, like a fish swimming in the same direction as a current.

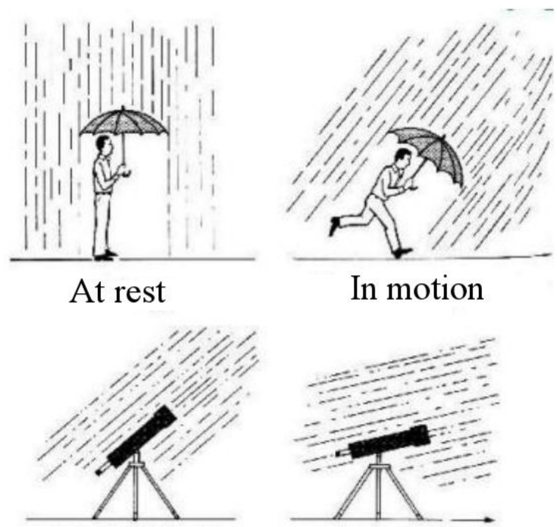
So, if you are at rest in ether, as it is the case for Bob here, the speed of light will appear to be the same in every direction. If you are moving in ether, as it is the case for Joe here, the speed is different depending on the direction. Notice that the difference between the speeds of light in opposite directions according to Joe is $1.1c - 0.9c = 0.2c$, which is twice the speed of ether according to Joe. This speed difference is always equal to twice the speed of ether.

Michelson-Morley Experiment

The previous reasoning led to the conception of an experiment to measure the speed of the Earth in ether. If the Earth is at rest in ether, the speed of light will be the same in every direction. If the Earth is moving through ether, the speed will be different depending on the direction. The difference between the largest measured speed and the smallest measured speed will be equal to twice the speed of the Earth in ether.

Michelson did this experiment in 1881 and did it again with Morley, with several improvements, in 1887. The 1887 experiment was so precise that a speed difference as low as 2 km/s speed could have been detected. To everyone's surprise, no variation in the speed of light was measured. If the speed of light is the same in all directions, then the Earth must be at rest in ether. This result could be explained by supposing that the ether revolved around the Sun at the same speed as the Earth, or by supposing that the Earth drags the ether along in its motion.

The problem was that other observations were showing that the Earth must be moving in ether. This is aberration. To understand this phenomenon, let's see what happens when it rains. If a person is at rest in the rain, this person should put the umbrella exactly above his head. If the person is running, he must tilt the umbrella since, in the person's reference frame, the rain now has a horizontal velocity.



supernovae.in2p3.fr/~llg/Enseignements/LP353/TD-correction-derniers-exos.pdf

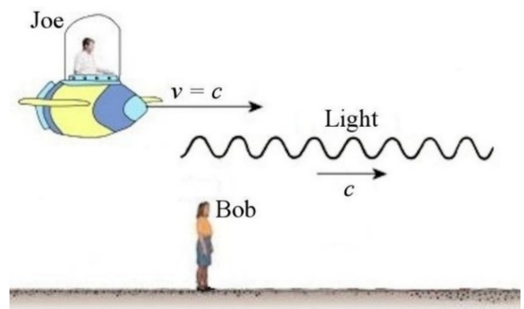
The same thing happens with light. If the Earth is at rest in ether, the light from a star arrives from a certain direction. If the Earth moves in ether. The light will come from a slightly different direction, which means that we have to change the alignment of telescopes a little bit to look at a specific star. This correction is only 0.005° at most, but it can be measured. As the direction of the Earth's velocity is constantly changing during the year, the alignment correction is constantly changing direction throughout the year. We can even calculate from these measurements that the Earth is moving at nearly 30 km/s around the Sun. According to the ether theory, there would be no aberration if the ether were to travel at the same speed as the Earth. This observation thus showed that the ether does not revolve around the Sun, or that it is not dragged by the motion of the Earth.

There is a contradiction. The Michelson-Morley experiment showed that the Earth is at rest in ether, and the aberration showed that the Earth is moving in ether. How can we get out of this paradox? This is the problem that physicists were trying to solve at the end of the 19^{th} century.

The Speed of Light in Vacuum Is Always the Same!

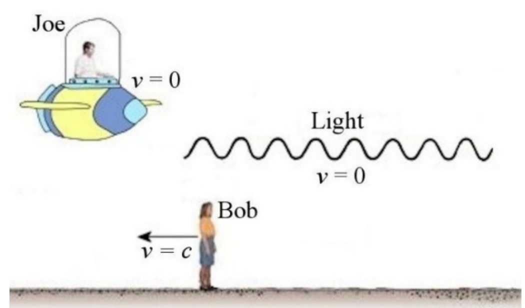
The speed of the Earth in ether was not the only problem. The equations of electromagnetism, obtained by Maxwell in 1865, are showing that light is an electromagnetic wave. When the speed of these waves is calculated, the result is, of course, the speed of light.

There was, however, a small conflict between these equations and relativity. The following situation illustrates this conflict. In this situation, there is an electromagnetic wave travelling towards the right. What will Joe observe if he travels at the speed of light?



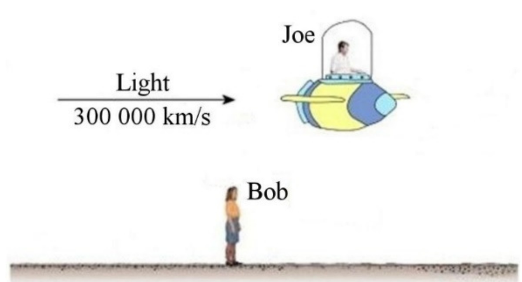
According to the Galilean equations of relativity, the speed of this beam of light according to Joe is

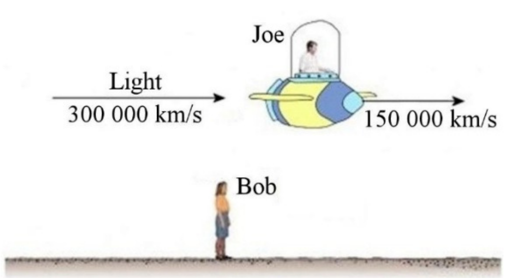
$$u_x' = u_x - v = c - c = 0$$



On the right, the situation in Joe's frame of reference is shown. Joe would, therefore, see an electromagnetic wave at rest. However, it is impossible to observe such a wave at rest according to Maxwell's equations. If these equations are solved, the result show that this wave must absolutely travel at 300 000 km/s according to Bob also! Obviously, there is a problem: Maxwell's equations did not comply with the equations of Galilean relativity. The speed of light in a vacuum should always be 300 000 km/s for all observers, regardless of their speed.

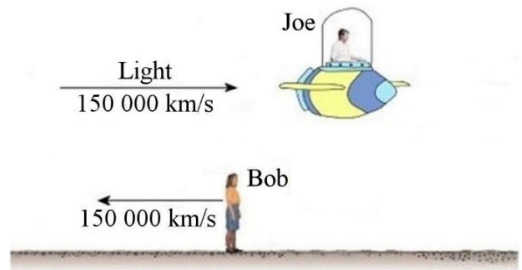
Let's take an example to illustrate this disagreement. Bob and Joe are both at rest, and a beam of light passes at the speed of light. Joe then measures the time it takes for the light to pass from the back of his ship to the front of his ship. Let's assume that the ship is 300 m long. In this case, it will take 1 μs for the light to pass from the back to the front of the ship.



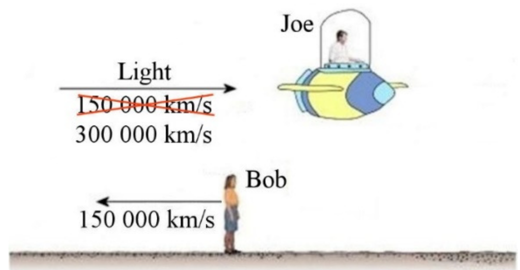


Joe then starts moving with his ship at a speed of $0.5c$.

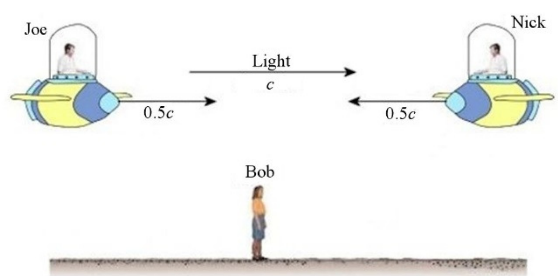
If we go to Joe's reference frame, we have the speeds shown on the right. According to Galileo's relativity, the beam of light only goes at 150 000 km/s according to Joe. At this speed, the time it takes for this light to travel the 300 m from the back of the ship to the front of the ship should then be 2 µs.



Yet, if Joe really does perform this experiment, he would measure a time of 1 µs, which means that the speed of the beam of light is not 150 000 km/s, but 300 000 km/s! This measurement is at odds with Galileo's relativity predictions! No matter how fast Joe is moving, he will always measure that the time taken by light to pass from one end of the ship to the other is always 1 µs!



Thus, in the situation shown on the right, all the observers will measure the same speed for the beam of light. This means that if Joe or Nick measures the time it takes for light to travel from one end of their ship to the other, they will always measure 1 µs if the ships are 300 m long. The measurement of the speed of the light beam according to Joe will give a value of c, and not $0.5c$ as predicted by the laws of transformation of velocities. The measurement of the speed of the light beam according to Nick will give a value of c, and not $1.5c$ as predicted by the laws of transformation of velocities. If you try to catch up with the light beam, as Joe does here, the beam will always be 300 000 km/s faster than you, whatever your speed. If you go towards the light, as does Nick here, it will always hit you at 300 000 km/s, whatever your speed.



http://www.youtube.com/watch?v=XR3OJwstfE8

9.4 EINSTEIN'S POSTULATES

So, there were some problems to solve. Does the Earth move in ether? How can Maxwell's equations and relativity be reconciled?

While some were trying to modify Maxwell's theory to be in agreement with Galileo's relativity, and others were trying to create a model of ether that could explain all the observations, Albert Einstein arrived, in 1905, at a completely different solution. Starting from 2 simple starting ideas (called *postulates*), he showed that these problems can be solved by modifying the equations of relativity.

Here are Einstein's 2 starting postulates.

Einstein's Postulates

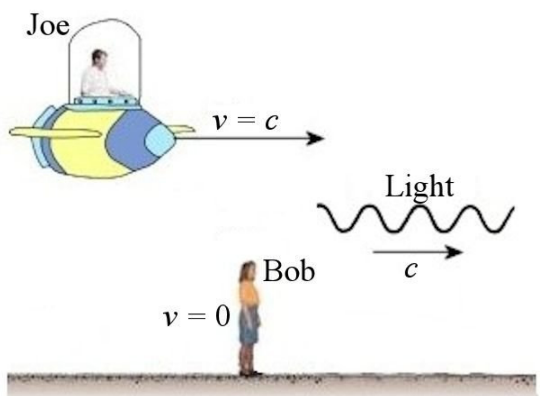
1) The laws of physics are the same for all the observers moving at a constant velocity.

2) The speed of light in a vacuum is always 300 000 km/s (actually 299 792.458 km/s) for all observers.

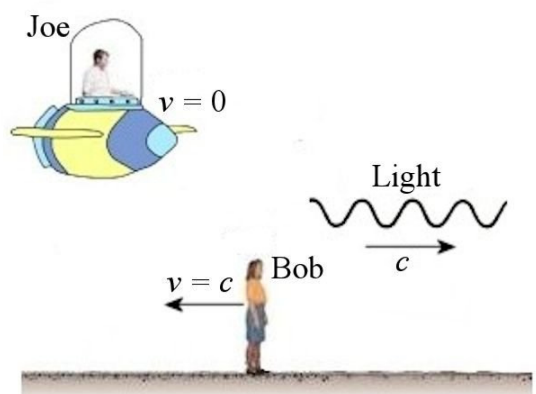
(Fun fact: the postulates of relativity say that the laws of physics and the speed of light are actually **not relative** quantities, i.e. dependent on the observer!)

The first postulate is simply the principle of relativity. However, Einstein went a little further than was accepted at the time. At the beginning of the 20th century, it was known that the principle is true for mechanics, but they didn't really know if the principle also applied to other phenomena such as optics and electromagnetism. Einstein postulates that this principle is true for all phenomena.

The second postulate explains why no difference in speed was measured in the Michelson-Morley experiment and why the same speed of light is always obtained with Maxwell's equations, regardless of the observer's speed. It may seem odd that several observers moving relative to each other all measure the same speed when they measure the speed of a light beam. This means that we have the following situation.



According to Bob



According to Joe

This is in flagrant contradiction with the equations of Galilean relativity.

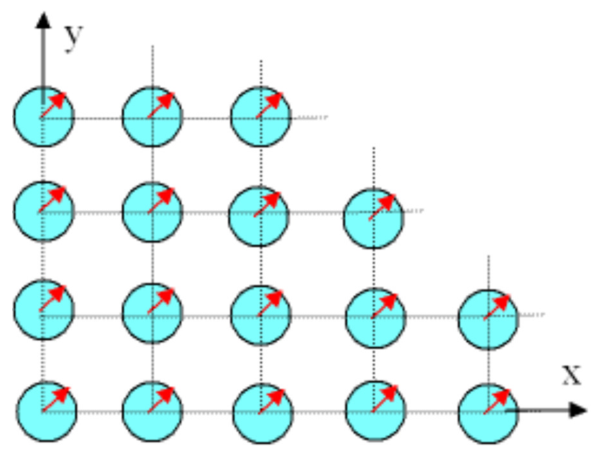
It is often mentioned that the postulates of relativity signed the death warrant of the ether. This is not entirely accurate. These postulates meant that the ether could still exist, but it had to keep exactly the same properties when we move from the point of view of one observer to the point of view of another. In other words, a speed cannot be attributed to the ether. The concept of velocity could not be applied to the ether. To try to build a model of the ether, it was therefore necessary to not give a specific velocity to the ether while giving it sometimes contradictory properties to explain all the observations (it had to be a rigid and elastic substance, which can penetrate matter without obstruction, which has no density, and which offers no resistance to the motion of objects…). The task was impossible, but it was not until the 1930's that the ether disappeared from physics.

Now, we will have to look at the consequences of these two postulates. Prepare to be flabbergasted, they are spectacular.

9.5 RELATIVITY OF TIME

The usual concept of time is completely shattered by Einstein's postulates. To understand why, some precision on how the observers note the time at which an event occurs must be given.

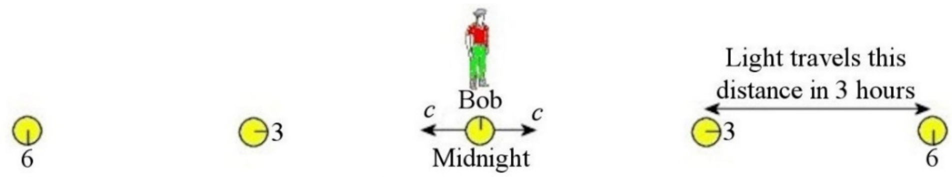
When an observer, say Bob, sees an event, he must calculate when the event really occurred, taking into account the time it took the light to reach him. If he sees a star exploding in 2017 and knows that the light from the explosion has taken 500 years to reach him, he will calculate that this explosion has truly occurred in 1517. To simplify this procedure, let's imagine that Bob has placed clocks throughout the universe (diagram). When an event occurs somewhere, the clock at this place prints a small piece of paper (a timestamp) or sends a signal to indicate the time. The observer then simply has to pick up the timestamps or the signals to know the time of the events.



www.pstcc.edu/departments/natural_behavioral_sciences/Web%20Physics/Chapter039.htm

For this solution to work, however, all the clocks must be synchronized, i.e. they must indicate the same time. Of course, Bob can synchronize its clocks at home and then go place them at their location in the universe, but it will be shown later that this solution is not good. (When the clocks are moved, the frame of reference changes and this can affect the time indicated.) Bob must, therefore, put each clock at its respective location before starting them. To start them, his central clock sends a light signal that starts the clocks when they receive it. We're using a light signal because we can easily know, by the second postulate, how such a signal behaves: it always travels at the speed c.

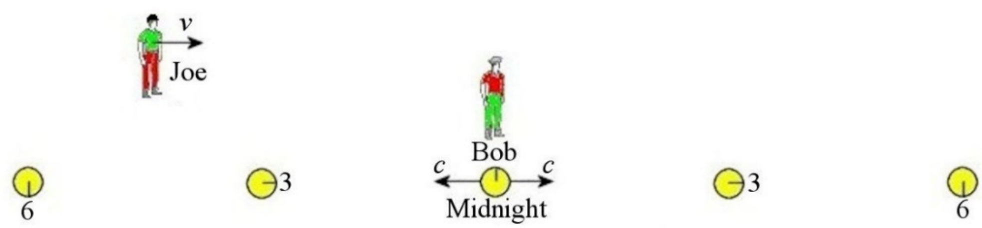
But Bob isn't stupid. He knows that it will take some time before the light signal reaches a specific clock. If all the clocks are set at midnight and Bob triggers the signal at midnight, a clock that receives this signal 3 hours later will start at 3 AM. The clock will then indicate midnight at 3 AM and will be 3 hours behind Bob's central clock. To compensate for this, Bob thought of not setting all the clocks at midnight. For example, he sets the clock that will receive the signal at 3 AM at 3 AM. When Bob sends its signal at midnight, this clock will start when the signal is received at 3 AM. As the clock was initially set at 3 AM, it will be synchronized with Bob's central clock.



Situation at midnight: departure of the signal

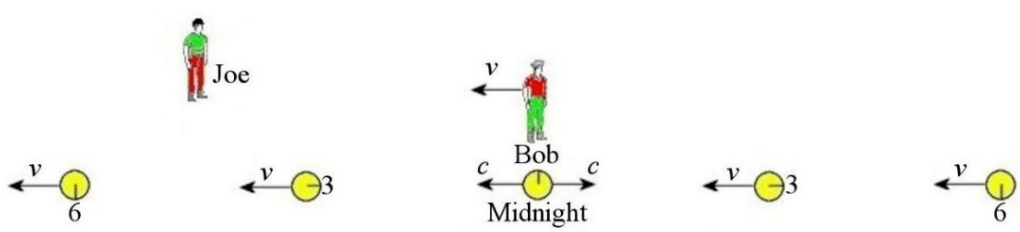
Situation at 6 h

Well done Bob, all your clocks are synchronized. There is, however, a problem: Let's see what happens according to another observer (Joe).



According to Bob

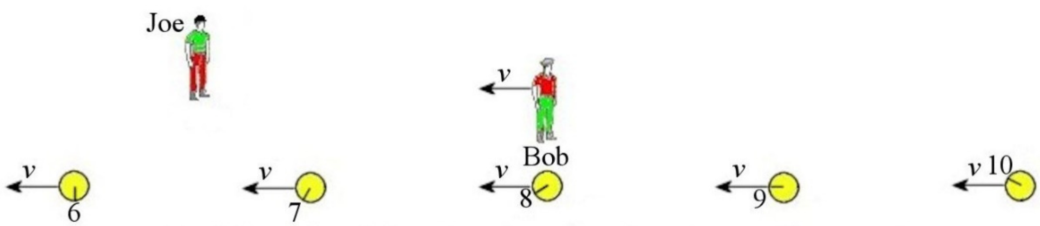
Joe sees Bob, with its clocks, moving towards the left at speed v.



According to Joe

But then, according to Joe, the clocks won't be synchronized. As clocks to the left of Bob are moving towards the left according to Joe, they are fleeing away from the light signal heading towards them. The light signal from Bob's central clock, however, does not travel faster as its speed remains c because the speed of light is the same for all observers. Therefore, it will take more time for the signal to reach these clocks because they are fleeing away from the signal, and they will start later than if they were at rest. If it takes

8 hours for the signal to catch up with a clock that was due to start at 6 AM, it will indicate 6 AM when Bob's central clock is indicating 8 AM. It is then 2 hours behind Bob's central clock. Thus, according to Joe, the clocks on the left are all behind Bob's central clock, and the farther away these clocks are from Bob, the larger is the time difference with Bob's central clock. For the clocks on the right of Bob, the opposite happens: they will start too early. These clocks are moving towards the light signal, and they will meet this signal sooner than if they were at rest. If it took only four hours for the signal to get to the clock which was supposed to be triggered at 6 AM, then this clock starts 2 h too soon so it will be ahead of Bob's central clock by 2 h. The clocks on the right are all ahead of Bob's central clock according to Joe, and the time difference gets larger as the clocks are farther away from Bob. Therefore, the result of the synchronization, according to Joe, is



End Result of the Synchronization According to Joe

In conclusion, Bob's clocks are not synchronized according to Joe, whereas they are synchronized according to Bob! Joe is going to tell Bob that he did not account for the fact that he and his clocks are moving when he made the synchronization. But Bob is going to reply, and this is correct, that it is Joe and not him who is moving, and that his synchronization is perfect. The problem is even bigger than that because Joe also has to install his clocks everywhere in the universe, and these clocks are moving at the same speed as Joe. He synchronizes them the same way and, by the same reasoning, we realize that Joe's clocks are synchronized according to Joe but are not according to Bob.

Who is right? Are the clocks synchronized or not? In fact, everyone is right. In relativity, the time at which an event occurred changes according to the observers. When the observers see the same explosion, they all calculate when the explosion truly happened taking into account the time it took the light to reach them. They then arrive at different conclusions. For example, Bob may infer that an event occurred in 1955 while Joe calculates that the same event occurred in 1969. To prove it, each may show the timestamp printed by the clock that each observer has installed at the place where the event occurred. Bob will have his timestamp showing 1955 and Joe would have his timestamp showing 1969. The crazy thing is that they are both right. The time at which an event occurs depends on the observer.

Time in Relativity

If an observer is moving with respect to another, the moments at which events occur is not the same for these two observers.

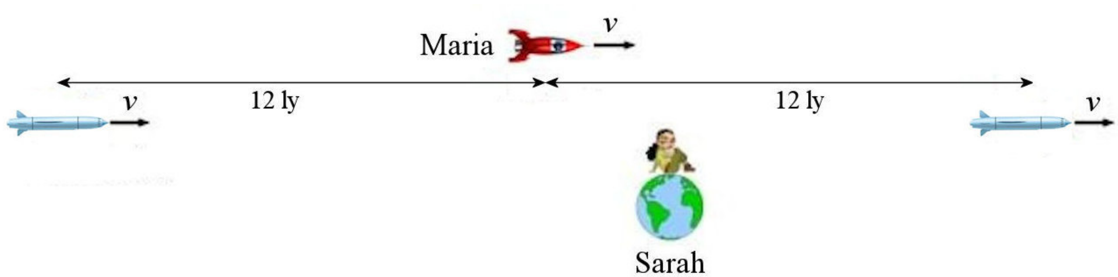
Of course, Joe will always be able to say that the information provided by Bob's clock is not good because Bob's clocks are not synchronized according to Joe. But Bob will then reply that his clocks are perfectly synchronized and that it is rather the information given by Joe's clock that isn't worth shit since Joe's clocks are not synchronized according to Bob… Joe will answer that his clocks are perfectly synchronized, and that the information is valid. And they are both right!

Already, it is obvious that Einstein's relativity will shake up your conception of time. The absolute time of Newton's physics and Galilean relativity does not exist anymore. In Newton's and Galileo's theories, an event occurring in 1929 occurs in 1929 for all observers. That is what absolute time means: everyone gets the same value. With Einstein's relativity, the event occurs at different times for each reference frame and this time depends on the speed of the observer. Therefore, time becomes relative.

9.6 SIMULTANEITY

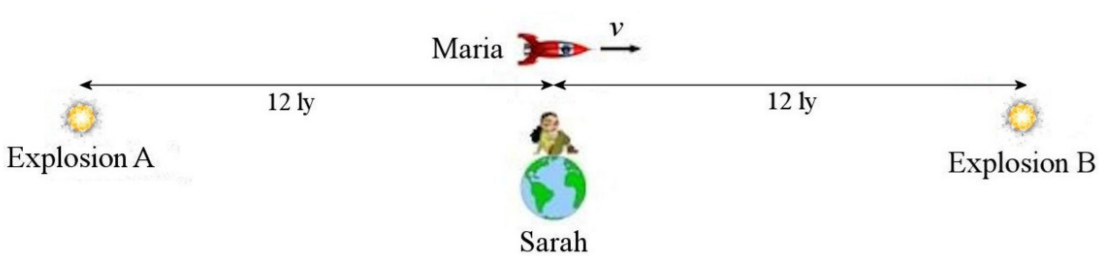
To illustrate once more how the notion of time changes, let's explore a situation where an observer sees two explosions at the same time.

Sarah is on Earth while Maria is travelling in a spaceship that goes very fast. Sarah observes that there are two missiles, which are also moving at speed v towards the right, and that Maria is halfway between the missiles, 12 ly from each missile.



www.how-to-draw-cartoons-online.com/cartoon-earth.html and fr.depositphotos.com/6046354/stock-photo-Spaceship.html and www.canstockphoto.ca/flying-cruise-missiles-20039121.html

Just at the moment when Maria passes beside the Earth, Sarah observes that the missiles explode. Let's assume that both their calendars indicate 2016 when they pass one beside the other.



As Sarah is equidistant from each explosion (12 light-years), the light of each explosion will take the same time to reach Sarah (12 years) and she will see the explosions at the same time (in 2028).
Knowing this, Sarah can infer what Maria will see. If the two explosions occurred at the same time, the light leaves from the two explosions at the same time. However, it will take more time for the light coming from explosion A to reach Maria because the light from this explosion and Maria are both going towards the right. The light, therefore, has to catch up with Maria, and it will take more time for the light of explosion A to get to Maria than to arrive at Earth. The opposite happens with explosion B. The light coming from explosion B goes towards the left, and Maria goes towards the right. Maria is going towards this light, and it will take less time for the light coming from explosion B to get to Maria than to arrive at Earth. Maria will, therefore, see explosion B before she sees explosion A. So far, no problem. An observer can see two events at the different time even if they occurred at the same time, it depends on the time taken by the light coming from each event to reach the observer.

If, according to Sarah, Maria sees explosion B before explosion A, it is certain that this is what Maria sees. This fact cannot be relative to the observer. The problem comes from what Maria will infer from this information. Let's take the point of view of Maria in this situation.

In Maria's reference frame, the spaceship is at rest and the Earth is moving towards the left. Knowing that Maria is halfway between the explosions, the light coming from the two explosions takes exactly the same time to reach Maria. But as she saw explosion B before explosion A, she is going to conclude that explosion B actually occurred before explosion A. If Sarah's speed is 0.6*c* according to Maria, calculations show that explosion B occurred in 2007 and explosion A in 2025. (This calculation will be made later in this chapter.)

Thus, the two explosions occurred at the same time (2016) according to Sarah while explosion B (in 2007) occurred before explosion A (2025) according to Maria. The following film shows a similar situation, but with a train.
http://www.youtube.com/watch?v=wteiuxyqtoM

So, who is right? Did the explosions occur simultaneously in 2016 as Sarah says or in 2007 (B explosion) and 2025 (explosion A) as Maria says? Both are right. In Sarah's reference frame, the two events are simultaneous and in Maria's reference frame, explosion B occurred before explosion A. Each could also provide the timestamps coming from their clocks to show that she is right. Sarah would have two timestamps indicating 2016 while Maria would have a timestamp showing 2007 and another showing 2025. Of course, they will both say that the clocks of the other are not synchronized and that the timestamps of the other mean nothing. Still, Sarah clocks are perfectly synchronized according to Sarah and Maria clocks are perfectly synchronized according to Maria and these timestamps truly indicate the time at which each explosion occurred according to each observer.

The same phenomenon comes up here: the moment at which an event occurs becomes relative to the observer. Here, the two events were simultaneous according to Sarah and the two events were not simultaneous according to Maria. Even if this is only one example, the following conclusion can be drawn (a formal proof will be done later).

Simultaneity

If two events are simultaneous for an observer A, they are not simultaneous for all the observers moving with respect to observer A.

A conclusion that was nearly drawn by Henri Poincaré in 1898. (He was close to discovering relativity before Einstein.)

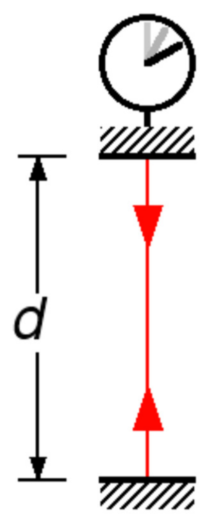
9.7 TIME DILATION

The Formula

It will now be shown that the rate at which time passes is also relative to the observer.

To illustrate this, a very peculiar clock is used. This clock works with a ray of light that is continuously reflected between two mirrors. Thus, the light hits the top mirror at regular intervals, say 1 millisecond. Whenever the light returns to the top mirror, the clock moves forward by 1 ms.

This special clock is used because the only thing we know for the moment is that the speed of light is the same for all the observers. By taking a clock working with light, we can easily find out what's happening for the other observers.

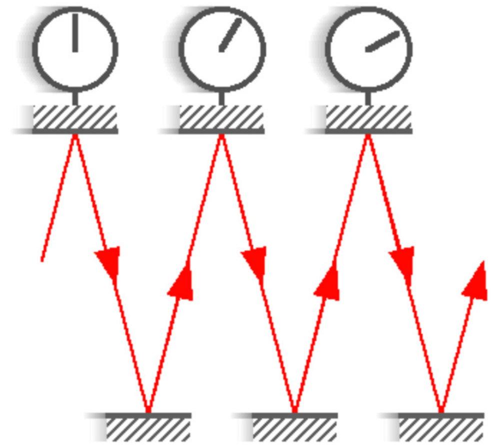


commons.wikimedia.org/wiki/File:Light-clock.png

The time it takes for the light to make a round trip in this clock when it is at rest is

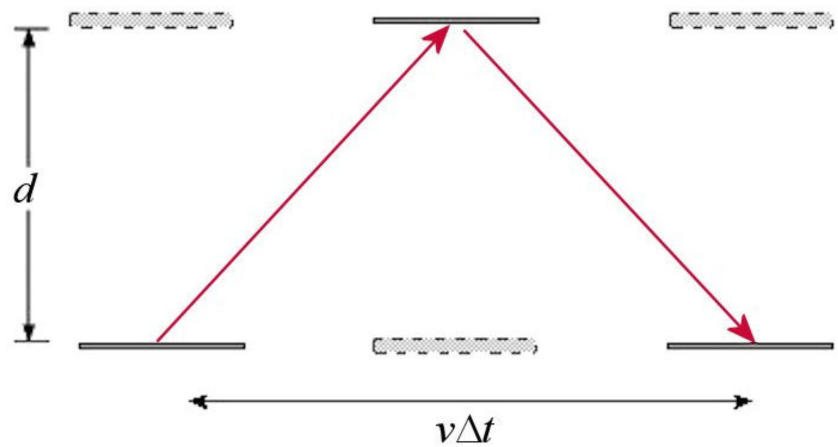
$$\Delta t_0 = \frac{2d}{c}$$

However, the time for this round trip will be longer if the clock is moving, It is easy to understand why this clock is going at a slower rate when it moves. In this case, the path of the light is illustrated in the diagram on the right. The vertical distance between the mirrors remained the same, but the light is now following a zigzag path. Each of these diagonal lines is longer than the distance between the mirrors so it will take more time to go from one mirror to another since the ray must travel a greater distance at the same speed.



commons.wikimedia.org/wiki/File:Light-clock.png

Let's calculate the time it takes for the light to make one round trip for an observer who sees the clock in motion.



www.astro.cornell.edu/academics/courses/astro201/time_dilation.htm

Let's say that the time is Δt to make one round trip according to this observer. During this time, the clock has moved forward a distance $v\Delta t$. Therefore, the length of a diagonal path (the hypotenuse) is

$$\sqrt{d^2 + \left(\frac{v\Delta t}{2}\right)^2}$$

Therefore, the time required to make one round trip is

$$\Delta t = \frac{2 \times hypotenuse}{c}$$

$$\Delta t = \frac{2\sqrt{d^2 + \left(\frac{v\Delta t}{2}\right)^2}}{c}$$

This equation must be solved for Δt. With a little algebra, it becomes

$$\left(\frac{c\Delta t}{2}\right)^2 = d^2 + \left(\frac{v\Delta t}{2}\right)^2$$

Since

$$\Delta t_0 = \frac{2d}{c}$$

the equation becomes

$$\left(\frac{c\Delta t}{2}\right)^2 = \left(\frac{c\Delta t_0}{2}\right)^2 + \left(\frac{v\Delta t}{2}\right)^2$$

$$\Delta t^2 = \Delta t_0^2 + \frac{v^2}{c^2}\Delta t^2$$

$$\Delta t^2 \left(1 - \frac{v^2}{c^2}\right) = \Delta t_0^2$$

The end result is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This square root comes up often in relativity, and the following symbol is used to represent this quantity.

γ Factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Therefore, the time dilation formula is

Time Dilation

$$\Delta t = \gamma \Delta t_0 = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

(v is the relative velocity between the observers.)

Note that the value of γ increase with speed, going from 1 (when $v = 0$) to ∞ (when $v = c$).

This formula was obtained for the first time by Einstein in 1905. It predicted that clocks in motion are moving at a slower rate than clocks at rest, i.e. that time flows at a slower rate when you are moving!

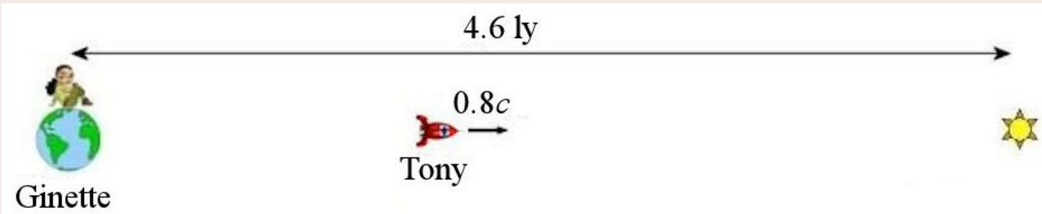
In fact, everything that changes over time is a clock and humans who age are like clocks. If clocks in motion are going at a slower rate, this means that humans will age less quickly when they are moving than if they remain at rest!

This movie gives you the same explanation.
http://www.youtube.com/watch?v=KHjpBjgIMVk

Let's go on with a small example.

Example 9.7.1

Ginette stays on Earth while Tony travels towards a star located 4.6 light-years away from Earth. The speed of Tony's spaceship is 80% of the speed of light.



a) What is the duration of the trip according to Ginette?

The duration of the trip according to Ginette is

$$\begin{aligned}\Delta t &= \frac{\text{distance}}{\text{speed}} \\ &= \frac{4.6ly}{0.8c} \\ &= \frac{4.6 years \cdot c}{0.8c} \\ &= \frac{4.6 years}{0.8} \\ &= 5.75 years\end{aligned}$$

(Note this trick for light-years distances: 4.6 ly = 4.6 years·c. It sometimes simplifies the calculations a lot.)

b) What is the duration of the trip according to Tony?

The time measured in the spaceship is Δt_0 (see explanation below). So, the time is

$$\begin{aligned}\Delta t_0 &= \Delta t\sqrt{1-\tfrac{v^2}{c^2}} \\ &= 5.75 years \cdot \sqrt{1-\tfrac{(0.8c)^2}{c^2}}\end{aligned}$$

$$= 5.75\,years \cdot \sqrt{1-0.8^2}$$
$$= 3.45\,years$$

Surprising, isn't it? The effect is even more dramatic if the speed is increased to 99.9% of the speed of light. The time according to Ginette is then 4.605 years and the time according to Tony is 0.206 year or approximately two and a half months. If Tony does a round trip to the star and comes back to Earth, 9.21 years will have elapsed on Earth, while a little less than 5 months will have passed for Tony. Ginette will then be 9.21 years older whereas Tony will be about 5 months older than when he left the Earth.

Let's push a bit further with a more sophisticated example. Suppose now that Tony goes to the centre of Galaxy (26 000 light-years away) in a spaceship. To go there, he accelerates at 9.8 m/s² for one half of the journey and then decelerates at 9.8 m/s² for the other half of the journey. These accelerations give an apparent weight identical to the weight on Earth to Tony. Once at the centre of the Galaxy, Tony comes back to Earth. Calculations (more complex than the ones made in the example since the speed changes constantly) show that the duration of this trip for Ginette is 52 004 years and only 39.6 years for Tony! An astronaut can make this trip during his lifetime. When Tony returns to Earth, he is about 40 years older than when he left but he doesn't recognize anybody on his return since 52 004 years have elapsed on Earth. It's not even sure that there is someone who remembers he was gone. Maybe apes would have taken control of the Earth by then…
(To see how this calculation was done, see this document:
http://physique.merici.ca/waves/dilation.pdf)

Is It Proven?

Is this effect proven or is it a mere fantasy of physicists? There are actually a lot of experimental proof that this effect is real. None implies sending astronauts on long trips since the current technology is not advanced enough to achieve such large speeds with a spaceship. However, in 1971, a clock was installed on a plane while another identical clock remained on the ground. The clocks were previously synchronized. After a small trip, the aircraft came back, and the clock in the plane was slightly behind the other clock (by 214 ns). The time gap between the two was exactly the one predicted by relativity.
http://www.youtube.com/watch?v=gdRmCqylsME

Even if an astronaut cannot be sent at speeds close to the speed of light, such large speeds can be achieved for small particles. This is what particle accelerators are for. In these accelerators, particles can go almost at the speed of light. However, some particles have a limited life span. For example, muons have an average lifetime of 2.2 µs when they are at rest. (Muons are particles very similar to electrons, but they are 207 times more massive.) When muons move at high speeds, they live much longer. This is exactly what relativity predicts: when they move, they age less quickly and therefore live longer for an observer who sees the particles move. The lengthening of the lifetime of a muon is exactly equal to the lengthening obtained with the time dilation formula.

Muons also provided one of the first proofs of relativity. When cosmic rays hit the atmosphere, they create, among other things, muons travelling at very high speeds. Even if the muons were travelling at the speed of light, they would travel only 660 m in 2.2 μs. However, these muons are formed at an altitude of a few tens of kilometres, and they still reach the surface of the Earth. The lengthening of their lifetime due to time dilation is what allows them to reach the ground. The number of muons reaching the ground is exactly as predicted by the theory of relativity.

It is possible to think that relativity plays no role in our daily lives, and this is not far from the truth. Be aware, however, that clocks in satellites would not tick at the same rate as those on Earth if they were to be placed next to each other. Once in orbit, they tick at the same rate as those on Earth because of relativistic effects. If engineers had not taken into account relativistic effects in the design of satellites, the satellite clocks would have slowly drifted compared to the clocks on Earth, which would have led to miscalculations. For example, there are clocks inside GPS satellites. If relativistic effects had not been taken into account, the GPS data could be off by up to 5 km, even if the clocks are resynchronized every 12 hours.

Which Observer Measures Δt_0?

In the previous example calculations, it is said that Tony measures Δt_0 which is called the *proper time*. Why is it Tony?

Precisely, proper time is

Proper Time Δt_0

Proper time is the time between two events according to an observer who observes the two events in the same place.

In most cases, this means that it is the time between the events according to the observer who is present at both events. In our example, Ginette is not present at the two events that mark the beginning of the trip (departure) and the end of the trip (the arrival). She is present when Tony leaves, but not at the arrival. However, Tony is present at the departure and at the arrival and it is him that measures the proper time.

It's not that obvious that Tony measures the proper time if the true definition of proper time is taken. It's clear that Ginette does not observe both events in the same place: the departure is close to the Earth and the arrival is near the star which is 4.6 ly away from the Earth. This is indisputably not in the same place. It is less straightforward for Tony. In order to see that the two events are in the same place for Tony, let's look at this situation in Tony's reference frame. According to Tony, he is at rest while the Earth recedes, and the star is coming towards him.

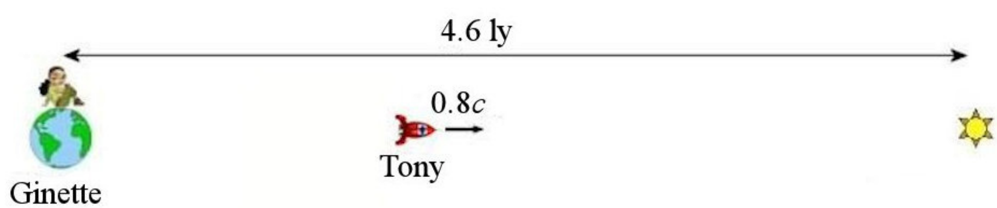
For Tony, the Earth is next to his ship at his departure and the star is next to his ship at arrival. For Tony, these two events are in the same place: next to his ship. It is, therefore, him that measures the proper time.

Common Mistake: Taking the Formula for Time Dilation When None of the Observers Measures Δt_0

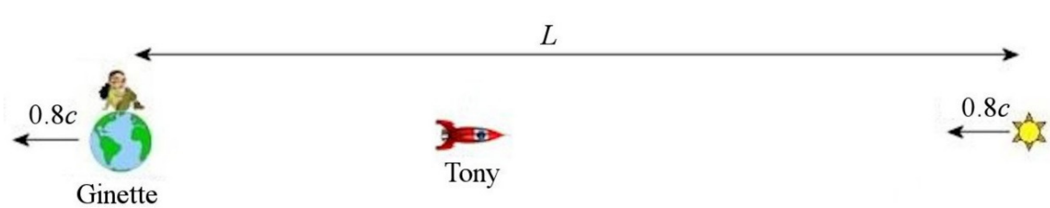
The time between two events is not necessarily Δt_0 for one of the two observers. It is possible to have two events at different places for both observers. In this case, do not use the time dilation formula. Use instead the Lorentz transformations (that will be seen later in this chapter).

9.8 LENGTH CONTRACTION

If time dilation were the only effect, there would be a problem of logic in relativity. Let's go back to the example of Ginette and Tony to illustrate this. According to Ginette, the situation is as follows.



According to previous calculations, the duration of the trip is 5.75 years for Ginette and 3.45 years for Tony. Let's look at this situation in Tony's reference frame to see what happens according to him.



In this reference frame, Tony is at rest, and the star is coming towards him. But if the star were 4.6 ly away and is moving towards Tony at $0.8c$, the time it would take for the star to arrive at Tony would be

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$t = \frac{4.6y \cdot c}{0.8c}$$

$$t = 5.75y$$

But the star is supposed to arrive in 3.45 years. In order to arrive at this value, there is only one solution: the star must be closer than 4.6 ly according to Tony. If the star arrives in 3.45 years, the initial distance must be

$$\begin{aligned} L &= v\Delta t_0 \\ &= 0.8c \cdot 3.45y \\ &= 2.76ly \end{aligned}$$

Thus, the distance between the Earth and the star is 4.6 ly according to Ginette but only 2.76 ly according to Tony. This is called *length contraction*.

The distance between two objects at rest is denoted L_0. This variable can also represent the length of an object at rest because the length of an object at rest is the distance between its two ends. L_0 is called the *proper length*.

The distance between two moving objects (at the same speed) is denoted L. This variable can also represent the length of a moving object.

The length contraction formula can be easily found by repeating the reasoning made in the example of Tony and Ginette. According to Ginette, Tony's speed is

$$v = \frac{L_0}{\Delta t}$$

According to Tony, the speed of the star (which is equal to Ginette's speed) is

$$v = \frac{L}{\Delta t_0}$$

As Tony's speed according to Ginette is the same as that Ginette's speed according to Tony, the following equation must hold.

$$\frac{L_0}{\Delta t} = \frac{L}{\Delta t_0}$$

$$L = L_0 \frac{\Delta t_0}{\Delta t}$$

$$L = L_0 \frac{\Delta t_0}{\Delta t_0 / \sqrt{1 - \frac{v^2}{c^2}}}$$

The end result is

Length Contraction

$$L = L_0\sqrt{1-\tfrac{v^2}{c^2}} = \frac{L_0}{\gamma}$$

(v is the relative velocity between the observers.)

This formula was obtained by FitzGerald in 1889 and, independently, by Lorentz in 1892. (They were then trying to explain the results of the Michelson-Morley experiment.) Thus, this contraction is sometimes called the *FitzGerald-Lorentz contraction*.

Let's check if the formula works. For Tony, the distance between the Earth and the star is

$$\begin{aligned}L &= L_0\sqrt{1-\tfrac{v^2}{c^2}}\\ &= 4.6ly\cdot\sqrt{1-\frac{(0.8c)^2}{c^2}}\\ &= 4.6ly\cdot\sqrt{1-(0.8)^2}\\ &= 2.76ly\end{aligned}$$

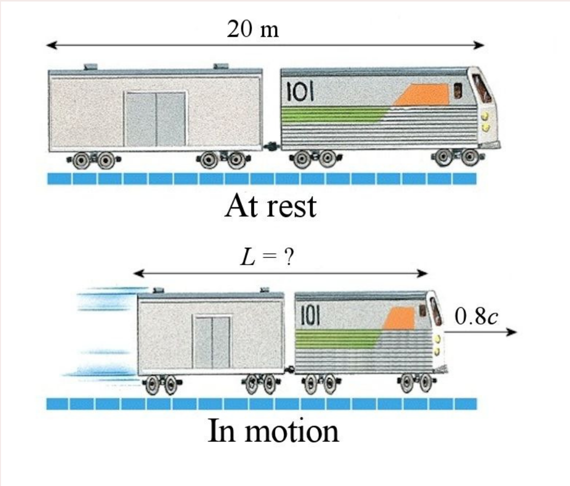
Bingo, it works

Example 9.8.1

A train is 20 m long at rest. What is its length if it is moving at $0.8c$?

The length is

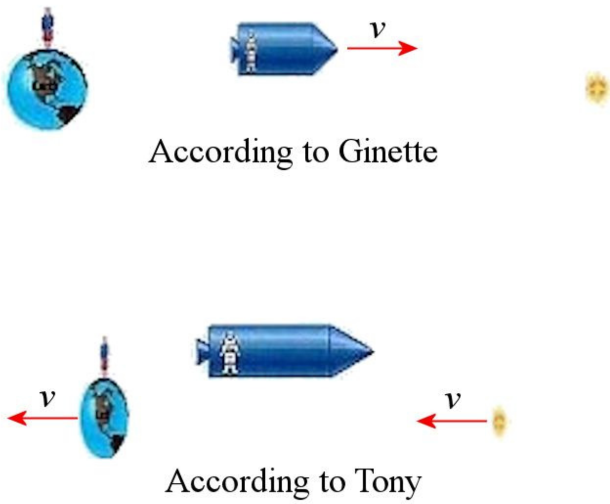
$$\begin{aligned}L &= L_0\sqrt{1-\tfrac{v^2}{c^2}}\\ &= 20m\cdot\sqrt{1-\frac{(0.8c)^2}{c^2}}\\ &= 20m\cdot\sqrt{1-(0.8)^2}\\ &= 12m\end{aligned}$$



www.zamandayolculuk.com/cetinbal/htmldosya1/SpecialTheory.htm

The only contracted dimension is in the direction of the velocity. The dimensions perpendicular to the velocity remain the same. Here, the train has the same height at rest and in motion. Only the length of the train changes.

The passengers of the train do not feel contracted. To illustrate this idea, let's take the example of Ginette and Tony again. In Ginette's frame, Tony's spaceship moves, and the spaceship is shorter than at rest. However, the Earth, Ginette and the star are at rest in this frame and are, therefore, not contracted.



kspark.kaist.ac.kr/Twin%20Paradox/Relativity%20Facts.htm

In Tony's frame, the spaceship and Tony are at rest and are, therefore, not contracted. For Tony, the Earth, Ginette, and the star are moving and are contracted.

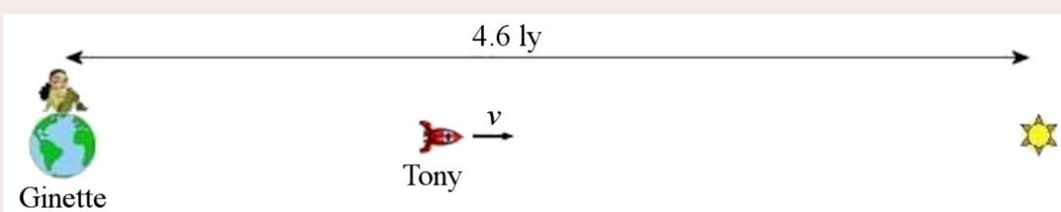
Thus, nobody sees himself contracted since each observer is at rest in its own frame. All the observers are observing that the other observers are contracted. The same thing happens with time dilation: all observers see themselves at rest and time runs normally for them. All observers are observing that the time passes more slowly for the other observers.

Example 9.8.2

Tony wants to make a trip from the Earth to a star located 4.6 ly away according to observers on Earth. What must be the speed of Tony for the trip to last 1 year according to Tony?

The equation to solve can be obtained by taking either point of view: Tony's or Ginette's (which is on Earth).

According to Ginette, we have the following situation.



The time according to Ginette is

$$\Delta t = \frac{L_0}{v}$$

Ginette can then calculate the time according to Tony with

$$\Delta t_0 = \Delta t\sqrt{1-\frac{v^2}{c^2}}$$
$$= \frac{L_0}{v}\sqrt{1-\frac{v^2}{c^2}}$$

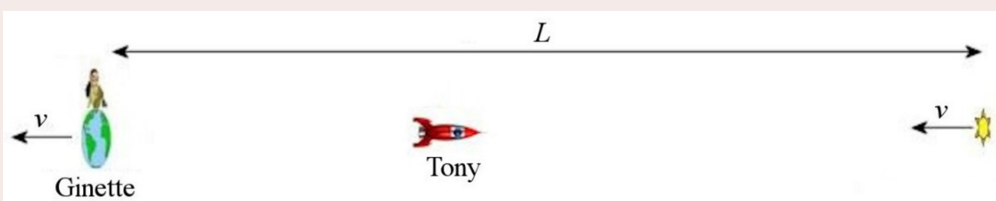
If the trip is to last one year, the following equation must be solved

$$1y = \frac{4.6y \cdot c}{v}\sqrt{1-\frac{v^2}{c^2}}$$

The solution is

$$\frac{1}{4.6} = \frac{c}{v}\sqrt{1-\frac{v^2}{c^2}}$$
$$\left(\frac{1}{4.6}\right)^2 = \frac{c^2}{v^2}\left(1-\frac{v^2}{c^2}\right)$$
$$\left(\frac{1}{4.6}\right)^2 = \frac{c^2}{v^2} - 1$$
$$\frac{1}{4.6^2} + 1 = \frac{c^2}{v^2}$$
$$\frac{v}{c} = \frac{1}{\sqrt{1+\frac{1}{4.6^2}}} = 0.97718$$

The speed must then be 97.718 % of the speed of light. Notice that the same equation to solve is obtained by taking the point of view of Tony. In Tony's reference frame, the situation is as follows.



According to Tony, the time it takes for the star to reach his spaceship is

$$\Delta t_0 = \frac{L}{v}$$

L is then found with the length contraction formula. The equation is now

$$\Delta t_0 = \frac{L_0\sqrt{1-\frac{v^2}{c^2}}}{v}$$

If the trip is to last 1 year, then the equation to solve is

$$1y = \frac{4.6a \cdot c \cdot \sqrt{1-\frac{v^2}{c^2}}}{v}$$

This is the same equation as the one obtained by taking the point of view of Ginette.

Effects at Low Speed

In our daily lives, the effects of time dilation and length contraction are rarely taken into account. But if you want to take these effects into account at low speeds, the following approximations make the calculations easier.

Approximations of the γ Factor if $v \ll c$

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1 + \frac{v^2}{2c^2} + \dots$$

$$\sqrt{1-\frac{v^2}{c^2}} \approx 1 - \frac{v^2}{2c^2} + \dots$$

Example 9.8.3

Pimprenelle stays in Quebec City while Amarante goes to Montreal (distance = 250 km) in a high-speed train travelling at 250 km/h.

a) What is the difference between the duration of the trip according to Pimprenelle and Amarante?

The duration of the trip according to Pimprenelle is

$$\Delta t = \frac{\text{distance}}{\text{speed}} = \frac{250km}{250\frac{km}{h}} = 1h = 3600s$$

The duration of the trip according to Amarante is

$$\begin{aligned}\Delta t_0 &= \Delta t\sqrt{1-\frac{v^2}{c^2}} \\ &= 3600s \cdot \sqrt{1-\frac{v^2}{c^2}}\end{aligned}$$

If this value is calculated with a calculator, 3600 s is obtained again because the speed is too small, and the answer is barely smaller than 3600 s. To see the time difference, an approximation is used.

$$\begin{aligned} difference &= \Delta t - \Delta t_0 \\ &= 3600s - 3600s \cdot \sqrt{1-\tfrac{v^2}{c^2}} \\ &= 3600s - 3600s \cdot \left(1 - \frac{v^2}{2c^2} + \ldots\right) \\ &= 3600s \cdot \frac{v^2}{2c^2} + \ldots \\ &= 3600s \cdot \frac{\left(69.44 \tfrac{m}{s}\right)^2}{2\left(3\times 10^8 \tfrac{m}{s}\right)^2} + \ldots \\ &\approx 9.645\times 10^{-11} s \end{aligned}$$

Amarante's watch is, therefore, behind Pimprenelle's watch by only 9.645×10^{-11} s after the trip to Montreal. The effect is not very important when the speed is far from the speed of light. A person who travels on a plane for 100 years would live only 1.5 ms longer than a person who had remained at rest on Earth.

b) What is the difference between the distance from Montreal to Quebec according to Pimprenelle and the distance from Montreal to Quebec according to Amarante?

The distance according to Pimprenelle is 250 km.

The distance according to Amarante is contracted. It is, therefore,

$$L = 250km\sqrt{1-\tfrac{v^2}{c^2}}$$

If this value is calculated with a calculator, 250 km is obtained again because the speed is too small, and the answer is barely smaller than 250 km. To see the difference in distance, an approximation is used.

$$\begin{aligned} \Delta L &= L_0 - L \\ &= 250km - 250km \cdot \sqrt{1-\tfrac{v^2}{c^2}} \\ &\approx 250km - 250km \cdot \left(1 - \tfrac{v^2}{2c^2} + \ldots\right) \\ &\approx 250km \cdot \frac{v^2}{2c^2} \\ &\approx 250km \cdot \frac{\left(69.44 \tfrac{m}{s}\right)^2}{2 \cdot \left(3\times 10^8 \tfrac{m}{s}\right)^2} \\ &\approx 6.69nm \end{aligned}$$

Length contraction makes the distance between Montreal and Quebec City only 6.69 nm shorter according to Amarante.

9.9 DOPPLER EFFECT WITH LIGHT (TAKE 2)

The Formula

The Doppler effect seen in previous chapters is modified by relativity because the rate at which time passes, related to the period of the wave, is changed by the speed of the source.

1) Classical Doppler Shift

The formula of the Doppler effect was

$$f' = \frac{v - v_0}{v - v_s} f$$

Since every reference frame is equivalent, the frame where the observer is at rest and the source is moving is always used. Therefore, the speed of the observer is always $v_0 = 0$. Since the speed of light is always c, we have $v = c$. The equation then becomes

$$f' = \frac{c}{c - v_s} f$$

The index s for the speed is then dropped since there is no possibility of confusion with other speeds. The frequency and wave period are thus

$$f' = \frac{c}{c - v} f \qquad \text{and} \qquad T' = \frac{c - v}{c} T$$

2) Time Dilation

However, if the source is moving, the period is changed compared to what we would have if the source were at rest. This means that

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where T_0 is the period of the source when it is at rest.

Relativistic Doppler Shift

Combining these two effects, we have

$$T' = \frac{c - v}{c} T$$

$$T' = \frac{c-v}{c}\frac{T_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

This becomes

$$T' = T_0 \frac{c-v}{\sqrt{c^2-v^2}}$$
$$T' = T_0 \frac{c-v}{\sqrt{(c-v)(c+v)}}$$
$$T' = T_0 \sqrt{\frac{c-v}{c+v}}$$

Therefore, the period and frequency shifts are

Relativistic Doppler Effect

$$T' = T_0 \sqrt{\frac{c-v}{c+v}}$$
$$f' = f_0 \sqrt{\frac{c+v}{c-v}}$$

(v is the speed of the source according to a motionless observer.)

The sign convention for speed stays the same as it was. The positive direction is from the source towards the observer. Thus, the speed of the source is positive if it is heading towards the observer and negative if it is moving away from the observer.

Remember that the point of view of the observer (observer at rest and source in motion) must always be taken to use this formula.

Example 9.9.1

A source at rest emits light with a 600 nm wavelength (orange). What is the wavelength seen by an observer who sees this source moving towards him at 30% of the speed of light?

The frequency of this wave when the source is at rest is

$$\begin{aligned} f_0 &= \frac{c}{\lambda_0} \\ &= \frac{3\times 10^8 \frac{m}{s}}{600\times 10^{-9} m} \\ &= 5\times 10^{14} Hz \end{aligned}$$

Therefore, the frequency received by the observer is

$$\begin{aligned} f' &= f_0\sqrt{\frac{c+v}{c-v}} \\ &= 5\times 10^{14}Hz\cdot\sqrt{\frac{c+0.3c}{c-0.3c}} \\ &= 5\times 10^{14}Hz\cdot\sqrt{\frac{1.3}{0.7}} \\ &= 6.814\times 10^{14}Hz \end{aligned}$$

The wavelength is thus

$$\begin{aligned} \lambda' &= \frac{c}{f'} \\ &= \frac{3\times 10^{8}\frac{m}{s}}{6.814\times 10^{14}Hz} \\ &= 440.3nm \end{aligned}$$

The source emits an orange light at rest. It is perceived as being violet by the observer who sees the source moving towards him at $0.3c$.

What the Observers See

The Doppler effect formula allows us to calculate what is **<u>seen</u>** by an observer.

Let’s take an example to illustrate.

We already know that time passes more slowly for Naomi according to Ophelia because of the time dilation. This is the result of the observations of Ophelia. Ophelia can calculate that for each second elapsed in Naomi’s spaceship, 1.25 seconds have elapsed on Earth.

$$\begin{aligned} \Delta t &= \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}} \\ &= \frac{1s}{\sqrt{1-(0.6)^2}} \\ &= 1.25s \end{aligned}$$

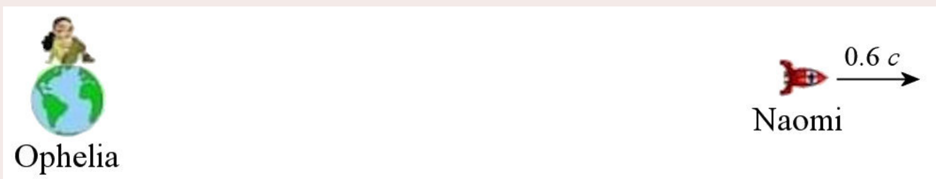
However, if Naomi's ship emits a flash of light every second that does not mean that Ophelia will see a flash every 1.25 seconds because the time taken for the light from each flash to arrive must be taken into account.

However, this is exactly what the Doppler effect formula does. It takes into account time dilation and the time taken by the waves to arrive at the observer. Thus, the time between the flashes as **seen** by Ophelia is

$$T' = T_0\sqrt{\frac{c-v}{c+v}}$$

Example 9.9.2

In Naomi's spaceship, there is a clock that emits a flash of light every second. What is the time between the flashes as **seen** by Ophelia if Naomi's spaceship moves away from Earth at 0.6*c*?



The time between the flashes is

$$\begin{aligned} T' &= T_0\sqrt{\frac{c-v}{c+v}} \\ &= 1s\cdot\sqrt{\frac{c-(-0.6c)}{c+(-0.6c)}} \\ &= 1s\cdot\sqrt{\frac{1.6}{0.4}} \\ &= 2s \end{aligned}$$

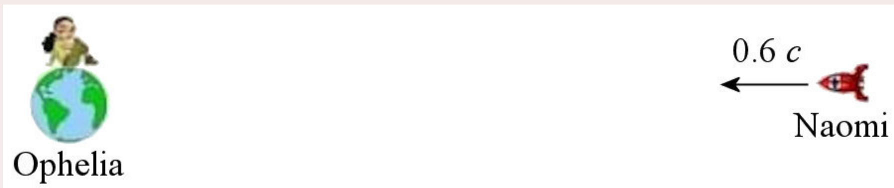
The speed is negative because the positive direction goes from the source towards the observer so from Naomi to Ophelia.

Ophelia sees a flash every 2 seconds. There is a difference between what is observed and what is seen. Ophelia **sees** a flash every 2 seconds, but she **observes** that the time between the flashes is 1.25 s. It takes more time to see the flashes, because the spaceship is farther and farther away at each flash and it will take more and more time for the light to reach Ophelia, thereby increasing the time between the flashes. At 0.6*c*, the spaceship moves, according to Ophelia, 2.25×10^8 m between each flash if there are 1.25 seconds between them. The light coming from the next flash must then travel this extra distance before arriving on Earth, and this takes 0.75 seconds. The time between the flashes as seen by Ophelia is, therefore, 2 seconds (1.25 s + 0.75 s).

It can, therefore, be concluded that Ophelia **sees** everything that happens in the spaceship at a rate 2 times slower than what is happening on Earth, like a movie playing in slow motion.

Example 9.9.3

In Naomi's spaceship, there is a clock that emits a flash of light every second. What is the time between the flashes as **seen** by Ophelia if Naomi's spaceship is moving towards Earth at 0.6*c*?



The time between the flashes is

$$\begin{aligned} T' &= T_0\sqrt{\frac{c-v}{c+v}} \\ &= 1s\cdot\sqrt{\frac{c-(0.6c)}{c+(0.6c)}} \\ &= 1s\cdot\sqrt{\frac{0.4}{1.6}} \\ &= 0.5s \end{aligned}$$

Ophelia **sees** a flash every 0.5 seconds but **observes** that the flashes are emitted every 1.25 seconds. It takes less time to see the flashes, because the spaceship is closer and closer to Earth at each flash and it will take less and less time for the light to arrive, thereby decreasing the time between the flashes. At 0.6*c*, the spaceship moves, according to Ophelia, 2.25×10^8 m between each flash if there are 1.25 seconds between them. The light coming from the next flash has less distance to travel before arriving on Earth, and this removes 0.75 seconds to the time. The time between the flashes as seen by Ophelia is, therefore, 0.5 seconds (1.25 s – 0.75 s).

It can, therefore, be concluded that Ophelia **sees** everything that happens in the spaceship at a rate 2 times faster than what is happening on Earth, like a movie playing in fast forward, even if the time is going at a slower rate in the spaceship!

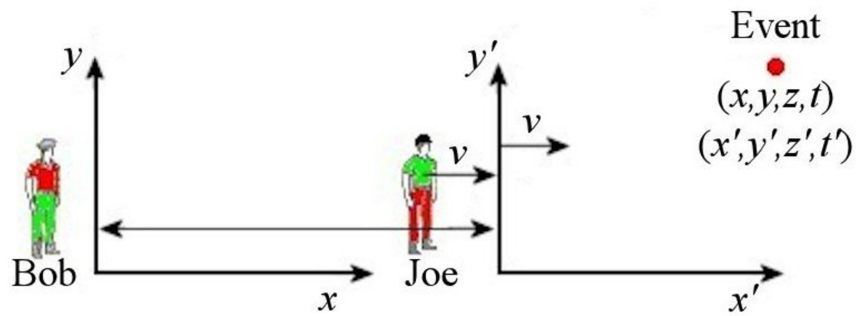
Here's a great video showing what is seen if someone is moving with speed close to the speed of light.
http://www.youtube.com/watch?v=JQnHTKZBTI4

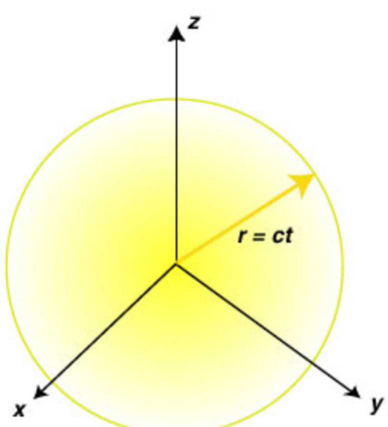
9.10 LORENTZ TRANSFORMATIONS

A more formal approach to relativity will be now used to obtain the transformation laws from one observer to another. With these transformations, all the formulas obtained previously will be proven again but more formally. More complex cases can also be dealt with using the transformation laws.

The Formulas

When two observers observe an event, they note the position and the time at which the event occurred with their own coordinate system.





www.zamandayolculuk.com/Cetinbal/HTMLdosya1/RelativityPrinciple.htm

To find out how to switch from one reference frame to another in accordance with Einstein's second postulate, let's imagine that a bright flash of light is spreading at speed c in every direction. The light is, therefore, creating a sphere with a radius that increases at the speed of light. As the equation of a sphere is

$$x^2 + y^2 + z^2 = R^2$$

and as the radius of the sphere increases at the speed of light, the equation of this sphere of light, according to Bob, is

$$x^2 + y^2 + z^2 = (ct)^2$$

Joe also observes an increasingly large light sphere growing at speed c since the speed of light is the same for every observer. Therefore, the equation of the sphere according to Joe must be

$$x'^2 + y'^2 + z'^2 = (ct')^2$$

With the transformation laws, the equation of the sphere according to Bob must transform into the equation of the sphere according to Joe. Let's first show that Galilean transformations fail to do this.

Starting from the equation of the sphere according to Joe, the transformation gives

$$x'^2 + y'^2 + z'^2 = (ct')^2$$
$$(x - vt)^2 + y^2 + z^2 = (ct)^2$$
$$x^2 - 2xvt + v^2t^2 + y^2 + z^2 = (ct)^2$$

Obviously, this is not the equation of the sphere according to Bob. On the other hand, this result is the first clue needed to obtain the correct transformations. There is a cross term with x and t (the second term) that must be eliminated. It can be eliminated if

$$t' = t + fx$$

where f is a value that will be determined in order to make the second term vanish. With this new transformation law, the transformation of the equation of the sphere is

$$x'^2 + y'^2 + z'^2 = (ct')^2$$
$$(x - vt)^2 + y^2 + z^2 = (c(t + fx))^2$$
$$x^2 - 2xvt + v^2t^2 + y^2 + z^2 = c^2t^2 + 2c^2xft + c^2f^2x^2$$

The second term on the left can now be cancelled out by the second term of right if

$$f = -\frac{v}{c^2}$$

Then, the equation becomes

$$x^2 - \cancel{2xvt} + v^2t^2 + y^2 + z^2 = c^2t^2 - \cancel{2vxt} + \frac{v^2}{c^2}x^2$$

This is not quite it. This result can be written as

$$x^2 + v^2t^2 + y^2 + z^2 = c^2t^2 + \frac{v^2}{c^2}x^2$$
$$x^2\left(1 - \frac{v^2}{c^2}\right) + y^2 + z^2 = c^2t^2\left(1 - \frac{v^2}{c^2}\right)$$

The correct result would have been obtained if x and t had been divided by $\sqrt{1-\frac{v^2}{c^2}}$. Then, the final result is

$$x^2 + y^2 + z^2 = c^2t^2$$

which is the equation of the sphere according to Bob. Thus, the transformation laws obtained for x' and t' are

$$x' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(x - vt)$$

$$t' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(t - \frac{vx}{c^2}\right)$$

The full transformation laws, called the *Lorentz transformations*, are

Lorentz Transformations (1)

$$x' = \gamma(x - vt) \qquad x = \gamma(x' + vt')$$

$$y' = y \qquad y = y'$$

$$z' = z \qquad z = z'$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \qquad t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

(v is the speed of an observer according to another observer.)

These transformations laws were obtained in 1898 by Lorentz and in 1899 by Larmor but they gave a much different interpretation to the time transformation. Poincaré was close to making a correct interpretation in 1904 but it was Einstein who interpreted correctly, in 1905, that the times of events are not the same for different observers and that there is no absolute time.

Note that if $v \ll c$, the Galilean transformations are obtained again. Thus, the results obtained with Galilean transformations are still valid, provided that the speed between the observers is small compared to the speed of light.

These laws of transformation are not that useful since the position $x = 0$ and the time $t = 0$ are somewhat arbitrary. It would be better to have the transformation laws of the distance (Δx, Δy, Δz) and the time (Δt) between the events.

The x-component of the distance and the time between 2 events are, according to Bob,

$$\Delta x = x_2 - x_1$$

$$\Delta t = t_2 - t_1$$

The distance between the events according to Joe is

$$\Delta x' = x_2' - x_1'$$

Using Lorentz transformations, we have

$$\Delta x' = \gamma(x_2 - vt_2) - \gamma(x_1 - vt_1)$$

$$\Delta x' = \gamma\left((x_2 - x_1) - v(t_2 - t_1)\right)$$

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$

An identical formula as the one we had for the Lorentz transformations is obtained, except that there are now Δ in front of variables. In fact, this is what always happens, and the transformations are

Lorentz Transformations (2)

$$\Delta x' = \gamma(\Delta x - v\Delta t) \qquad \Delta x = \gamma(\Delta x' + v\Delta t')$$
$$\Delta y' = \Delta y \qquad \Delta y = \Delta y'$$
$$\Delta z' = \Delta z \qquad \Delta z = \Delta z'$$
$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \qquad \Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)$$

(v is the speed of an observer according to another observer.)

This form is a bit more useful because it does not depend on the arbitrary $x = 0$ and $t = 0$.

Before using these formulas, remember the following rule:

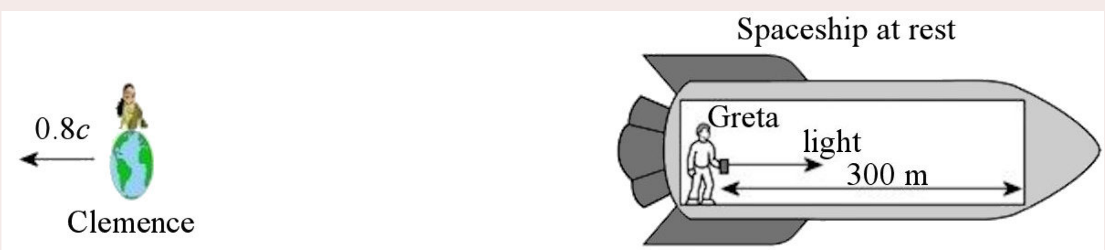
Which Observer Uses Primes?

The observer who sees the other travelling towards the negative x-axis uses primes

Here are some examples of the application of these formulas.

Example 9.10.1

Greta is in a spaceship moving away from Earth at 0.8c. She then sends a beam of light from the back towards the front of the ship. According to Greta, the distance between the light source and the target is 300 m. Clemence, on Earth, observe Greta's spaceship.



theoryoftime.com/wordpress/?p=236

a) How long does it take for the light to get to the target according to Greta?

The time is simply the time it takes for a beam of light to travel 300 m.

$$\Delta t' = \frac{300m}{3\times 10^8 \frac{m}{s}} = 10^{-6} s$$

Greta is the one who sees the other observer moving towards the negative x-axis. That's why Greta notes the time with a prime.

b) How long does it take for the light to get to the target according to Clemence?

The situation, according to Clemence, is

The time can be calculated with the Lorentz transformations.

The time according to Greta is $\Delta t' = 10^{-6}$ s, and the distance between the events (departure and arrival of the light) is, according to Greta, $\Delta x' = 300$ m.

Therefore, the time according to Clemence is

$$\begin{aligned}\Delta t &= \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)\\ &= \frac{1}{\sqrt{1-\frac{(0.8c)^2}{c^2}}}\cdot\left(10^{-6}s + \frac{0.8c\cdot 300m}{c^2}\right)\\ &= \frac{1}{\sqrt{1-0.8^2}}\cdot\left(10^{-6}s + \frac{0.8\cdot 300m}{c}\right)\\ &= 3\times 10^{-6}s\end{aligned}$$

So, it takes 3 μs for the light to arrive at the target according to Clemence while it takes only 1 μs according to Greta.

Note that this problem cannot be done with the time dilation formula because none of these observers measures the proper time (Δt_0) because none of the observers is present at both events. Better formulation: the two events (departure and arrival of the light) are not in the same place for any of these two observers, so none measures the proper time.

On the other hand, here's how this problem could have been handled without using Lorentz transformations. For Clemence, the distance between the starting point of the light and the target is smaller because of length contraction. For her, the length of the spaceship is

$$\begin{aligned}L &= 300m\cdot\sqrt{1-0.8^2}\\ &= 180m\end{aligned}$$

The light, travelling at c, must catch up with a target at 180 m which moves away at 0.8c. Therefore, the time to catch up is

$$\Delta t = \frac{180m}{c - 0.8c}$$
$$= 3\times 10^{-6} s$$

Simultaneity

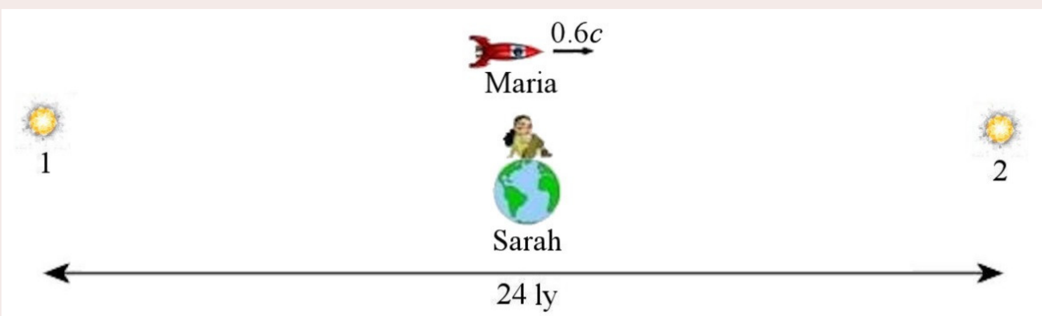
Lorentz transformations will confirm that two events that are simultaneous for an observer are not simultaneous for the other observers. If two events are simultaneous, then $\Delta t = 0$. The time, according to another observer, is therefore

$$\Delta t' = \gamma\left(\cancel{\Delta t} - \frac{v\Delta x}{c^2}\right)$$
$$\Delta t' = -\gamma\frac{v\Delta x}{c^2}$$

which cannot be equal to zero. This shows that the events cannot be simultaneous.

Example 9.10.2

Sarah observes that two missiles 24 ly one from the other (according to Sarah) explode simultaneously. What is the time between the explosions according to Maria?



According to Sarah, the time between events is Δt = 0 (since the events are simultaneous) and the distance between the events is Δx = 24 ly. So, the time between the events according to Maria is

$$\Delta t' = \gamma\left(\cancel{\Delta t} - \frac{v\Delta x}{c^2}\right)$$
$$= -\gamma\frac{v\Delta x}{c^2}$$

$$
\begin{aligned}
&= -\frac{1}{\sqrt{1-\frac{(0.6c)^2}{c^2}}} \cdot \frac{0.6c \cdot 24y \cdot c}{c^2} \\
&= -\frac{1}{\sqrt{1-0.6^2}} \cdot 0.6 \cdot 24y \\
&= -18\,years
\end{aligned}
$$

There is an 18-year gap between the explosions according to Maria. The sign of this answer indicates which event happened first. As we set $\Delta x = x_2 - x_1 = 24$ ly as positive, it means that event 2 is the explosion to the right. As a negative $\Delta t' = t'_2 - t'_1$ was obtained, t'_2 must be smaller than t'_1 ($t'_2 < t'_1$). This means that the event 2 occurred before the event 1. So, the explosion 2 occurred 18 years before explosion 1 according to Maria. (This is exactly the same situation we had when we first talked about simultaneity. We then said that Maria observes that the events occur in 2007 and 2025. The gap is effectively 18 years.)

Is it possible to find the years of these explosions according to Maria, knowing that the two explosions occurred in 2016 (according to Sarah), just as Maria was passing next to the Earth? Of course, but to calculate them, the position and the time of each event according to Sarah must be found first.

This year 2016 corresponds to the time $t = 0$, since $t = 0$ corresponds to the time when the two observers are at the same place. Thus, the two 2016 explosions occurred at $t = 0$ according to Sarah.

As Sarah is at $x = 0$ (the observer is always at the origin of its coordinates), one of the explosions occurred at $x = -12$ ly and the other at $x = 12$ ly.

Therefore, the time of the explosion 1 at $x = -12$ ly according to Maria is

$$
\begin{aligned}
t'_1 &= \gamma\left(t_1 - \frac{vx_1}{c^2}\right) \\
&= \frac{1}{\sqrt{1-\frac{(0.6c)^2}{c^2}}} \cdot \left(0 - \frac{0.6c \cdot (-12y \cdot c)}{c^2}\right) \\
&= \frac{1}{\sqrt{1-0.6^2}} \cdot \left(0 + 0.6 \cdot (-12y)\right) \\
&= 9\,years
\end{aligned}
$$

As $t' = 0$ is 2016, $t' = 9$ years is 2025.

The time of the explosion 2 at $x = 12$ ly according to Maria is

$$\begin{aligned} t_2' &= \gamma\left(t_2 - \frac{vx_2}{c^2}\right) \\ &= \frac{1}{\sqrt{1-\frac{(0.6c)^2}{c^2}}}\cdot\left(0-\frac{0.6c\cdot 12y\cdot c}{c^2}\right) \\ &= \frac{1}{\sqrt{1-0.6^2}}\cdot\left(0-0.6\cdot 12y\right) \\ &= -9\,years \end{aligned}$$

As $t' = 0$ is 2016, $t' = -9$ years is 2007.

(Correctly, the observer must not necessarily be at $x = 0$ of its reference system. The origin can be put anywhere. However, if the position of the origin is changed, remember that the $t = 0$ is the moment where the origins of the two axes systems of the observers are at the same place.)

Time Dilation

Δt_0 is the time between two events occurring at the same place according to one observer. So, for this observer, we have

$$\Delta t = \Delta t_0$$
$$\Delta x = 0$$

So, for another observer, the time is

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)$$
$$\Delta t' = \gamma\left(\Delta t_0 - \frac{v\cdot 0}{c^2}\right)$$
$$\Delta t' = \gamma\Delta t_0$$

which the time dilation formula.

Length Contraction

Measuring the length of an object consist of two separate events: measuring the position of an end of the object and measuring the position of the other end of the object. The subtraction of the two positions gives the length of the object ($L = x_2 - x_1$). When the object is in motion, the position of both ends must be measured simultaneously. If the position of the back end of the object is measured at one moment and the position of the front end of the object is measured 1 second later, the subtraction of the two positions does not give the length of the object in motion because the position of the front end of the object has changed during this time. The position of both ends has to be measured at the same time. Thus, for an observer who sees the object moving, we must have

$$L = x_2 - x_1 = \Delta x$$
$$\Delta t = 0$$

On the other hand, the position of the ends can be measured at different times if the object is at rest.

$$L_0 = x_2' - x_1' = \Delta x'$$
$$\Delta t' = \text{whatever}$$

The Lorentz transformations then give

$$\Delta x' = \gamma\left(\Delta x - v\Delta t\right)$$
$$L_0 = \gamma\left(L - v \cdot 0\right)$$
$$L = \frac{L_0}{\gamma}$$

which is the length contraction formula.

The Interval

The interval between two events is defined by

Interval

$$I = c^2\left(\Delta t\right)^2 - \left(\Delta x\right)^2 - \left(\Delta y\right)^2 - \left(\Delta z\right)^2$$

Note that *I* can be positive, negative or zero.

The interval is interesting because it is a relativistic invariant. This means that all the observers obtain to the same value for the interval.

Here is the proof that the interval is the same for everyone. If the observer with primes calculates the interval between two events, then

$$\begin{aligned} I' &= c^2\left(\Delta t'\right)^2 - \left(\Delta x'\right)^2 - \left(\Delta y'\right)^2 - \left(\Delta z'\right)^2 \\ &= c^2\gamma^2\left(\Delta t - \frac{v\Delta x}{c^2}\right)^2 - \gamma^2\left(\Delta x - v\Delta t\right)^2 - \left(\Delta y\right)^2 - \left(\Delta z\right)^2 \\ &= \gamma^2\left(c^2\left(\Delta t\right)^2 - \cancel{2v\Delta x\Delta t} - \frac{v^2}{c^2}\left(\Delta x\right)^2\right) \\ &\qquad - \gamma^2\left(\left(\Delta x\right)^2 - \cancel{2v\Delta x\Delta t} + v^2\left(\Delta t\right)^2\right) - \left(\Delta y\right)^2 - \left(\Delta z\right)^2 \end{aligned}$$

$$= \gamma^2 \left(c^2 (\Delta t)^2 \left(1 - \frac{v^2}{c^2}\right) - (\Delta x)^2 \left(1 - \frac{v^2}{c^2}\right)\right) - (\Delta y)^2 - (\Delta z)^2$$
$$= c^2 (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$
$$= I$$

The observer using primes therefore arrives at the same value as the observer who does not use primes. So, we have

Invariance of the Interval

$$I' = I$$

There are not a lot of things on which different observers agree, but they agree on the value of the interval.

Proper Time

As mentioned previously, the proper time Δt_0 between 2 events is the time between the events for the observer who observes the 2 events in the same place (so who could be present at the 2 events). In this case, the interval is

$$\begin{aligned} I &= c^2 (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \\ &= c^2 (\Delta t_0)^2 - (0)^2 - (0)^2 - (0)^2 \\ &= c^2 (\Delta t_0)^2 \end{aligned}$$

Thus, we arrive at the following formula.

Proper Time Between Two Events

$$\Delta t_0 = \tfrac{1}{c}\sqrt{I}$$

The different observers can therefore easily calculate the proper time from *I*.

If *I* is negative, there is no proper time. This happens when the two events are too far from each other so that no observer can observe them at the same place. To be present at the two events, the observer would have to travel faster than light and this is impossible (as will be seen later).

If *I* is positive, there is a proper time and the interval is then said to be *time-like*. In this case, the order of the events can never be reversed. All observers will see both events in the same order. This information can be useful for determining the sign of the proper time when the square root is calculated to find Δt_0 (same sign as the sign of Δt used to calculate *I*). Note also that if *I* is positive, no observer can observe that these events are simultaneous.

Proper Distance

The proper distance $\Delta\sigma$ between 2 events is the distance between the events for the observer who observes that the 2 events are simultaneous. The interval is then

$$\begin{aligned} I &= c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \\ &= c^2(0)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \\ &= -(\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \\ &= -(\Delta\sigma)^2 \end{aligned}$$

Thus, we arrive at the following formula.

Proper Distance Between Two Events

$$\Delta\sigma = \sqrt{-I}$$

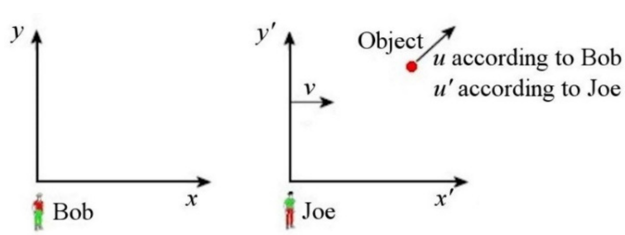
The different observers can therefore easily calculate the proper distance from *I*.

If *I* is positive, there is no proper distance. This occurs when the time between events is too great so that no observer can observe the events simultaneously.

If *I* is negative, there is a proper distance, and the interval is then said to be *space-like*. In this case, the order of the events in time can be reversed. Some observers observe event A before event B and some other will observe event B before event A. On the other hand, the order of events in space cannot change. If the value of *x* of event 1 is greater than the *x* of event 2 for an observer, it will be larger for all observers. This information can be useful for determining the sign of the proper distance when the square root is calculated to find $\Delta\sigma$ (same sign as the sign of Δx used to calculate *I* if $\Delta y = 0$ and $\Delta z = 0$, which is usually the case here). Note that if *I* is negative, no observer will be able to observe the two events at the same position

9.11 VELOCITY TRANSFORMATIONS

Now, two observers measure the speed of an object. The speed of this object is u according to one observer, and u' according to the other observer.



To know the speed of an object, the position of the object at two different instants must be measured. These are two events. With the distance between the events according to Bob (Δx) and the time between the events according to Bob (Δt), the *x*-component of the velocity of the object according to Bob is obtained.

$$u_x = \frac{\Delta x}{\Delta t}$$

With the distance between the events according to Joe ($\Delta x'$) and the time between the events according to Joe ($\Delta t'$), the x-component of the velocity of the object according to Joe is obtained.

$$u'_x = \frac{\Delta x'}{\Delta t'}$$

The transformation law of the x-component of the velocity of the object can be obtained with Lorentz transformations.

$$u_x = \frac{\Delta x}{\Delta t}$$

$$u_x = \frac{\cancel{\gamma}\left(\Delta x' + v\Delta t'\right)}{\cancel{\gamma}\left(\Delta t' + v\Delta x' / c^2\right)}$$

By dividing the numerator and the denominator by $\Delta t'$, the equation becomes

$$u_x = \frac{\left(\frac{\Delta x'}{\Delta t'} + v\right)}{\left(1 + \frac{v\Delta x'}{c^2\Delta t'}\right)}$$

Since

$$u'_x = \frac{\Delta x'}{\Delta t'}$$

the equation is

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}$$

This is the transformation law for the x-component of the velocity. By doing the same thing with the other components, the transformation laws for the velocities are obtained.

Velocities Transformations

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \qquad u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}$$

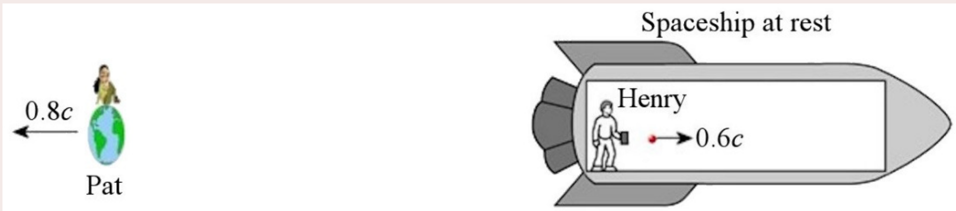
$$u'_y = \frac{u_y\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{vu_x}{c^2}} \qquad u_y = \frac{u'_y\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{vu'_x}{c^2}}$$

$$u'_z = \frac{u_z\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{vu_x}{c^2}} \qquad u_z = \frac{u'_z\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{vu'_x}{c^2}}$$

(v is the speed of an observer according to another observer.)

Example 9.11.1

In his spaceship, Henry throws a ball. The speed of the ball according to Henry is 0.6*c*. What is the speed of the ball according to Pat, who is on Earth?



According to Henry, the velocity is $u'_x = 0.6c$. According to Pat, the velocity of the ball speed is thus

$$\begin{aligned} u_x &= \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}} \\ &= \frac{0.6c + 0.8c}{1 + \frac{0.8c \cdot 0.6c}{c^2}} \\ &= \frac{0.6c + 0.8c}{1 + 0.8 \cdot 0.6} \\ &= 0.946c \end{aligned}$$

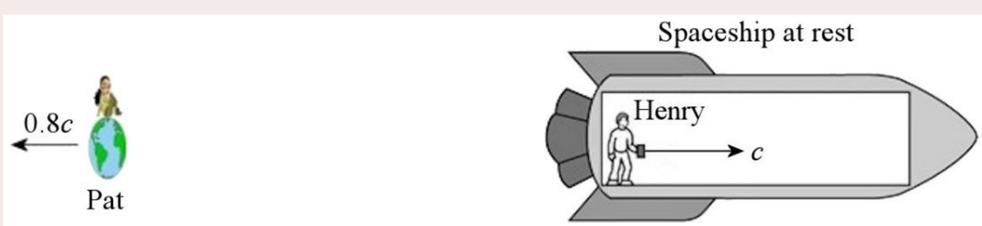
So, this is the situation according to Pat.

(Note that in Galilean relativity, the speed would have been 1.4*c* according to Pat.)

It's time to test the theory. What happens if Henri sends a beam of light towards the front?

Example 9.11.2

In his spaceship, Henry sends a beam of light towards the front of the ship. What is the speed of light according to Pat, who is on Earth?



According to Henry, the velocity is $u'_x = c$. According to Pat, the speed of light is thus

$$\begin{aligned} u_x &= \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}} \\ &= \frac{c + 0.8c}{1 + \frac{0.8c \cdot c}{c^2}} \\ &= \frac{c + 0.8c}{1 + 0.8} \\ &= c \end{aligned}$$

So, this is the situation according to Pat.

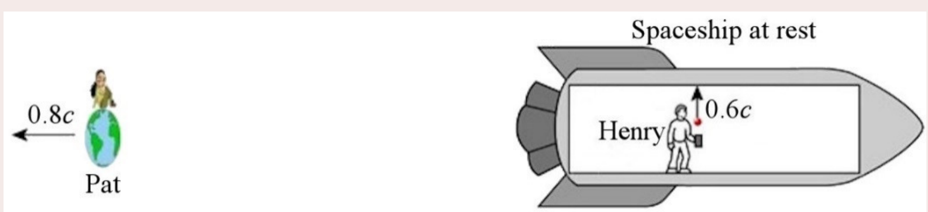
This result agrees with Einstein's second postulate.

(Note that in Galilean relativity, the speed of light would have been 1.8c according to Pat.)

Now, Henry will throw the ball upwards.

Example 9.11.3

In his spaceship, Henry throws a ball upwards. The speed of the ball according to Henry is 0.6c. What is the velocity of the ball according to Pat, who is on Earth?



According to Henry, the components of the velocity of the ball are $u'_x = 0$ and $u'_y = 0.6c$. The x-component of the velocity of the ball according to Pat is thus

$$\begin{aligned} u_x &= \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}} \\ &= \frac{0.8c}{1 + \frac{0.8c \cdot 0}{c^2}} \\ &= \frac{0.8c}{1 + 0} \end{aligned}$$

$$= 0.8c$$

It is just normal to have the same velocity as Henri's spaceship since the ball is following the motion of the spaceship.

The y-component of the velocity of the ball according to Pat is

$$\begin{aligned} u_y &= \frac{u'_y \sqrt{1-\frac{v^2}{c^2}}}{1+\frac{vu'_x}{c^2}} \\ &= \frac{0.6c \cdot \sqrt{1-\frac{(0.8c)^2}{c^2}}}{1+\frac{0.8c \cdot 0}{c^2}} \\ &= \frac{0.6c \cdot \sqrt{1-0.8^2}}{1+0} \\ &= 0.36c \end{aligned}$$

So, the speed of the ball according to Pat is

$$\begin{aligned} u &= \sqrt{u_x^2 + u_y^2} \\ &= \sqrt{(0.8c)^2 + (0.36c)^2} \\ &= 0.877c \end{aligned}$$

and its direction is

$$\begin{aligned} \theta &= \arctan \frac{u_y}{u_x} \\ &= \arctan \frac{0.36c}{0.8c} \\ &= 24.2° \end{aligned}$$

So, this is the situation according to Pat.

Example 9.11.4

On Earth, Florence sees Sam's and Alex's spaceships heading towards each other with a speed of $0.7c$. What is the speed of Sam's spaceship according to Alex?

Even if there are three observers in this problem, Florence and Alex are the only interesting observers because we are going to pass from Florence's point of view to Alex's point of view. As the speed between Florence and Alex is 0.7c, we have $v = 0.7c$ (v is always the speed between the observers when we pass from the point of view of one observer to the point of view of another. This speed is never negative.)

Each of these observers measures the velocity of the same object (Sam's spaceship here). The velocity of that object is u_x according to Florence and u'_x according to Alex (u_x and u'_x always refer to the velocity of the same object but according to different observers). Note that the values of u_x and u'_x are negative if the object is moving towards the negative x-axis according to the observer. This is what is happening here with Sam's spaceship. It is possible for u_x and u'_x to have different signs.

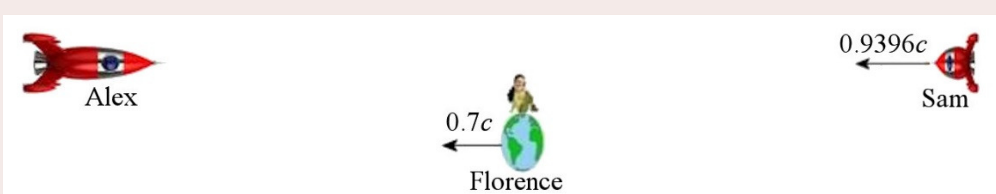
Therefore,

$$u_x = -0.7c$$
$$u'_x = \text{what we're looking for}$$
$$v = 0.7c$$

The velocity according to Alex is

$$\begin{aligned} u'_x &= \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \\ &= \frac{-0.7c - 0.7c}{1 - \frac{(0.7c)\cdot(-0.7c)}{c^2}} \\ &= \frac{-0.7c - 0.7c}{1 + 0.7 \cdot 0.7} \\ &= -0.9396c \end{aligned}$$

So, this is the situation according to Alex.



If Sam's point of view had been taken, Alex's spaceship would be moving towards Sam with a speed of 0.9396c.

Note that the speeds obtained will never exceed the speed of light. This is normal since it will be shown a little further in theses notes that the speed of an object can never exceed the speed of light.

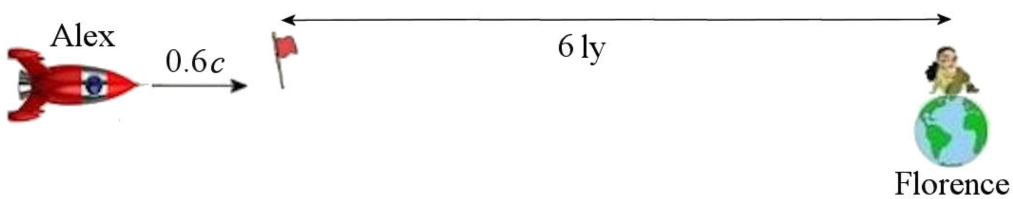
9.12 SOME FALSE PARADOXES

With relativity, it is quite easy to find situations that seem paradoxical at first glance. This means that different results are obtained by taking the point of view of different observers whereas different results should have been impossible. Very early in the history of relativity, some have highlighted such situations. The most famous is probably Paul Langevin's twin paradox of 1911 (Langevin is also famous for his extramarital affair with Marie Curie…). This paradox is a bit too complex to be addressed here because there is an observer that changes speed. However, simpler situations that seem to lead right to a paradox can be examined.

Florence and Alex

The first case concerns time dilation.

Florence, who is on Earth, has placed a flag 6 ly from the Earth. When Alex passes next to the flag, Alex and Florence both start a stopwatch.



When Alex will pass near the Earth, Alex and Florence will show each other their stopwatch. Alex and Florence will then be able to compare the values given by the stopwatches to see which is ahead and which is late.

According to Florence

Here is what is going to happen according to Florence. Between the moment Alex starts her stopwatch and the time she arrives at the Earth 10 years will have elapsed according to Florence (6 ly at 0.6*c*). However, Florence knows that Alex's stopwatch is ticking more slowly because it is moving. According to the time dilation formula, the stopwatch will advance by only 8 years while 10 years will have elapsed on Earth. Thus, according to Florence, Alex's stopwatch will indicate 8 years at the time of the meeting, while Florence's stopwatch, which is ticking at a normal pace, will indicate 10 years.

According to Alex

A paradox seems to appear if the point of view of Alex is taken. According to Alex, the situation is as follows.

According to Alex, the distance between the Earth and the flag is 4.8 ly according to the length contraction formula. Alex starts her stopwatch when the flag passes by her ship and she will stop it when the Earth passes by her. As the Earth is moving at 0.6c and is initially 4.8 ly away from Alex, the Earth will arrive in 8 years. Thus, Alex's stopwatch will indicate 8 years at the meeting, just as Florence had predicted.

However, Florence's stopwatch is ticking more slowly (since it is moving) according to Alex. As Florence's stopwatch is ticking more slowly, it should be late compared to Alex's stopwatch when they will meet. In fact, according to the time dilation formula, Florence's stopwatch will have advance by

$$8y = \frac{\Delta t_0}{\sqrt{1-0.6^2}}$$
$$\Delta t_0 = 6.4y$$

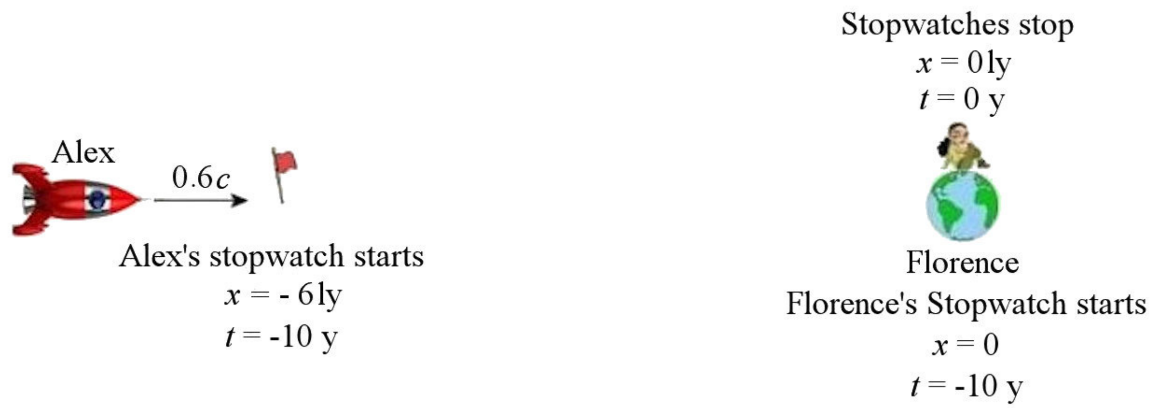
Thus, Florence's stopwatch should indicate 6.4 years at the meeting according to Alex.

The Apparent Paradox

There seems to be a paradox because they cannot be both right. What will be seen on Florence's stopwatch when they will meet? Are they going to see 10 years as Florence has predicted or will they see 4.8 years as Alex has predicted? They cannot be both right. If someone takes a picture of Florence's stopwatch when they will meet, the display of Florence's stopwatch can be seen on the photograph. It is impossible to have a result different from what is seen on the photograph.

The Solution

In reality, there is no paradox when the problem is done correctly. Florence's reasoning is quite correct: Florence's stopwatch will indicate 10 years at the meeting. It is the passage from Florence's reference frame to Alex's that was done incorrectly. To do it correctly, the positions and the time of the events according to Florence must be used.



The t = 0 is when the observers meet (this is what was assumed when the Lorentz transformations were obtained). Thus, the start time of the stopwatches is t = –10 years since they start 10 years before the meeting (which is the t = 0) according to Florence. As for the x = 0, it is set where Florence is (which is at the origin of her axes system). Lorentz transformations are then used to obtain the times and positions in Alex's reference frame.

According to Alex, the coordinates of the meeting (stopwatches stop) are

$$x'_R = \gamma\left(x_R - vt_R\right) = \frac{5}{4}\cdot\left(0-0\right) = 0$$

$$t'_R = \gamma\left(t_R - \frac{vx_R}{c^2}\right) = \frac{5}{4}\cdot\left(0-\frac{0.6c\cdot 0}{c^2}\right) = 0$$

(These values make sense since the time of the meeting is, by definition, t' = 0 and t = 0. In addition, the meeting is at the place where Alex is located at that time, which is the origin of Alex's axes.)

According to Alex, the coordinates of the start of Alex's stopwatch (Alex's stopwatch starts) are

$$x'_A = \gamma\left(x_A - vt_A\right) = \frac{5}{4}\cdot\left(-6y\cdot c - 0.6c\cdot\left(-10y\right)\right) = 0ly$$

$$t'_A = \gamma\left(t_A - \frac{vx_A}{c^2}\right) = \frac{5}{4}\cdot\left(-10y - \frac{0.6c\cdot\left(-6y\cdot c\right)}{c^2}\right) = -8y$$

This means that Alex's stopwatch starts 8 years before the meeting (which is at t = 0). Alex's stopwatch should, therefore, indicate 8 years when they will meet according to Alex, as expected by Florence and Alex.

According to Alex, the coordinates of the start of Florence's stopwatch (Florence's stopwatch starts) are

$$x'_F = \gamma(x_F - vt_F) \qquad t'_F = \gamma\left(t_F - \frac{vx_F}{c^2}\right)$$

$$= \frac{5}{4}\cdot\left(0 - 0.6c\cdot(-10y)\right) \qquad = \frac{5}{4}\cdot\left(-10y - \frac{0.6c\cdot 0}{c^2}\right)$$

$$= 7.5ly \qquad = -12.5y$$

Oh surprise! Florence's stopwatch will run for 12.5 years as it will stop at $t' = 0$. But this stopwatch is moving at $0.6c$ and is, therefore, ticking more slowly. The time dilation formula can be used to calculate by how much this stopwatch will advance.

$$12,5y = \frac{\Delta t_0}{\sqrt{1-0.6^2}}$$

$$\Delta t_0 = 10y$$

Florence's stopwatch should, therefore, indicate 10 years at the meeting according to Alex. Alex's forecasts are in line with those of Florence.

The error that led to the paradox was to think that the starts of the stopwatches were also simultaneously according to Alex. If the starts are simultaneous according to Florence, then they cannot be simultaneous according to Alex. This is what the Lorentz transformations are telling us: Alex starts her stopwatch at $t' = -8$ years and Florence starts her stopwatch at $t' = -12.5$ years. Florence started her stopwatch 4.5 years before Alex according to Alex. It is true that Florence's stopwatch is ticking more slowly according to Alex, but it will still indicate a greater value than Alex's stopwatch because it has started 4.5 years earlier than Alex's stopwatch.

Therefore, there is no paradox.

The transformations also indicate that Florence was 7.5 light-years away from Alex when she started her stopwatch ($x' = 7.5$ ly for the start of Florence's chronometer and Alex is always at $x' = 0$). As Florence moves at $0.6c$ according to Alex, she will arrive 12.5 years after the start of Florence's stopwatch, as expected.

Another Apparent Paradox: What If Alex's Stopwatch Emitted Flashes of Light?

Another apparent paradox can be obtained with this situation if we assumed that Alex's stopwatch emits a flash of light every year from the moment it starts. Since 8 years pass in the ship according to our calculations, the stopwatch should emit 8 flashes in total. Thus, Florence should see 8 flashes.

Now let's use the Doppler effect formula to calculate the time between flashes as seen by Florence.

$$\begin{aligned} T' &= T_0\sqrt{\frac{c-v}{c+v}} \\ &= 1y\cdot\sqrt{\frac{c-0.6c}{c+0.6c}} \\ &= 1y\cdot\sqrt{\frac{0.4}{1.6}} \\ &= 0.5y \end{aligned}$$

Florence sees a flash every 6 months. Since it takes 10 years for Alex to arrive according to Florence, one could conclude that Florence will see 20 flashes of light. However, we had come to the conclusion that the stopwatch would only emit 8 flashes of light. How can Florence see 20 flashes if the stopwatch has only emitted 8 flashes? Again, there seems to be a paradox.

We're going to use dates to help us resolve this paradox. Let's assume that, according to Florence, Alex's ship passes by the flag in 2020 and Alex passes by Earth in 2030. This results in a journey that lasts 10 years according to Florence, which is in line with what had been said. However, Florence doesn't immediately see the first flash emitted by Alex's stopwatch. Since the flag is 6 ly away, Florence will see this flash 6 years after Alex actually passes by the flag. She will therefore see the first flash in 2026. As Alex passes by Earth in 2030, Florence will see flashes for only 4 years (between 2026 and 2030). During these 4 years, she sees a flash every 6 months, for a total of 8 flashes. That's exactly the number of flashes emitted. There is no paradox.

Another Apparent Paradox: Alex Seems to Move Faster Than Light

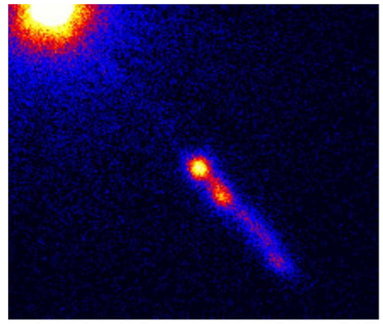
In this situation, Florence will see Alex go faster than light. She sees Alex passing by the flag in 2026 (6 years after Alex actually passed by the flag) and she sees Alex passing close to Earth in 2030. Florence has therefore seen Alex travel 6 ly (distance between the flag and the Earth) in 4 years (time between 2026 and 2030). With this distance and time, the speed is 1.5*c*.

But we'll see in the next section that an object cannot go faster than light. Again, there is an apparent paradox. How can Alex move at 1.5*c* if nothing can exceed the speed of light?

Even if Florence sees Alex going faster than light, it doesn't necessarily mean that Alex is really moving faster than light. In fact, Florence knows that she saw the start of the journey 6 years late and that the ship's journey really began in 2020. Thus, she observes that Alex has travelled 6 ly in 10 years and that his true speed is 6 ly/10 y = 0.6*c*.

This apparent speed greater than the speed of light is only possible if the object is moving towards the observer.

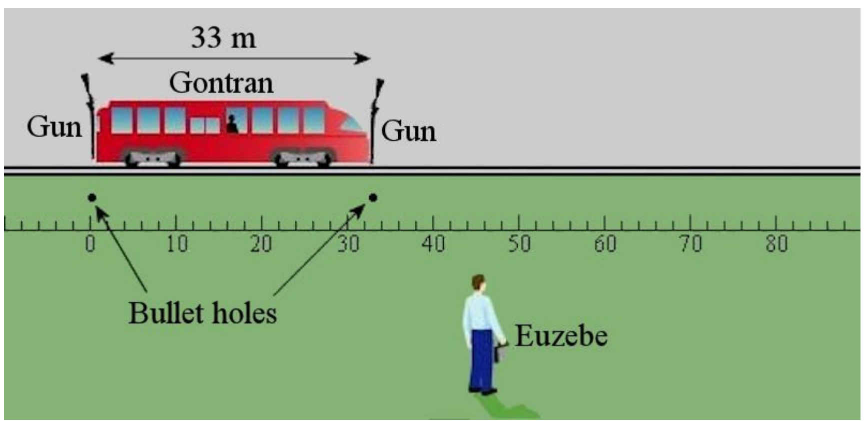
For example, the jet of gas made by quasar 3C 273 appears to have such an apparent speed greater than the speed of light. According to what we <u>see</u>, these jets seem to be moving towards us at a speed of almost 10*c* (we even have measurements up to 15*c*). However, it is <u>observed</u> that the true velocity is smaller than *c* (it is close to 0.99c).



en.wikipedia.org/wiki/3C_273

Guns in a Train

Now, here is a situation that also seems paradoxical but involving length contraction this time. Gontran is in a 33 m long train (length at rest) while Euzebe is on the ground and watches the train passing. Gontran has placed a gun at each end of the train. When he presses a button, the two guns shoot a bullet at the same time in the ground. Euzebe drew a ruler on the ground to measure the position of the bullet holes. Here is this situation when the train is at rest.



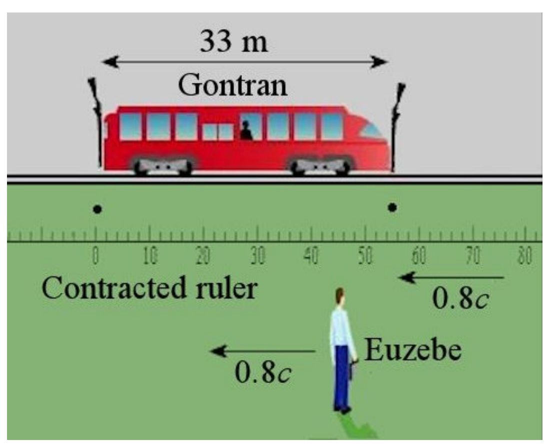
prancer.physics.louisville.edu/astrowiki/index.php/Special_Relativity

Now let's calculate the position of the bullet holes on Euzebe's ruler when the train is moving at 0.8*c*. Gontran's instructions are to fire when the gun located at the back of the train is above the 0 of the ruler drawn by Euzebe.

<u>According to Gontran</u>

In Gontran's reference frame, Euzebe and the ground are moving.

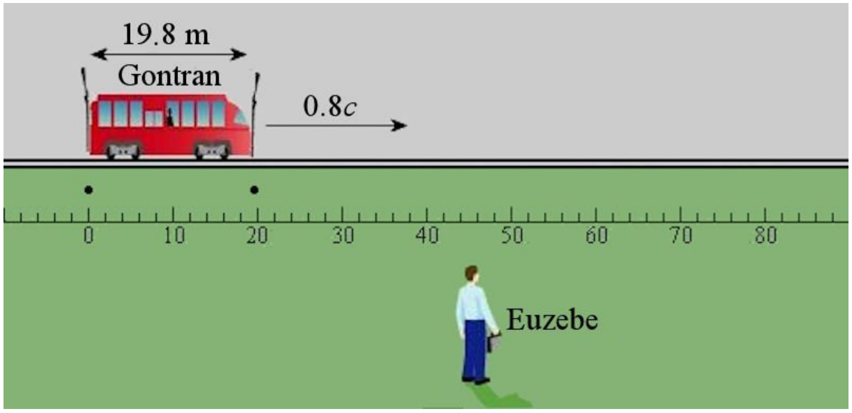
The guns are 33 m apart according to Gontran. However, the ground, moving at 0.8*c* according to Gontran, is contracted. The ruler, which was drawn on the ground, is also contracted so that the gun in front of the train is not above the 33 m mark when it fires.



According to the length contraction formula, the gun is above the 55 m mark when it fires. Therefore, it makes a hole in the ground at the 55 m mark on Euzebe's ruler.

According to Euzebe

This is the same situation as viewed from Euzebe's reference frame.



According to Euzebe, the train is contracted, and the guns are only 19.8 m apart according to the length contraction formula. When the guns are fired, the gun located in front of the train is above the 19.8 m mark. This gun will make a hole in the ground at the 19.8 m mark on Euzebe's ruler.

The Apparent Paradox

There is a paradox: the observers do not have the same position on Euzebe's ruler for the hole made by the bullet fired by the gun in front of the train. The hole should be at the 55 m mark according to Gontran and the 19.8 m mark according to Euzebe. Once again, a photo of the hole and of the marks of the ruler drawn by Euzebe on the ground cannot be different for each observer.

The Solution

There is obviously something that has not been done properly. To find out what went wrong, Lorentz transformations are used again. There are two events here: the firing of each gun. According to Gontran, the guns shoot simultaneously and are 33 m apart. Therefore,

$$\Delta x' = x_2' - x_1' = 33m$$
$$\Delta t' = 0$$

Giving a positive value to $x'_2 - x'_1$, automatically means that the firing of the gun in front of the train is event 2 and that the firing of the gun at the back of the train is event 1.

According to Lorentz transformations, the distance, and the time between the events according to Euzebe are

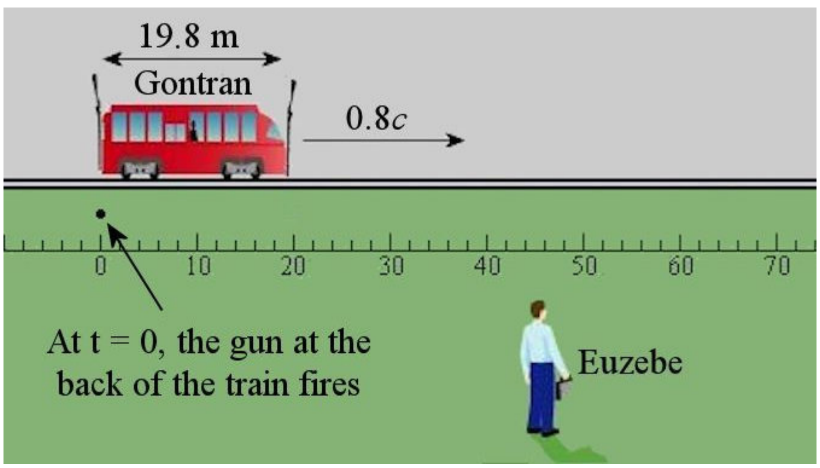
$$\Delta x = \gamma(\Delta x' + v\Delta t') \qquad \Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)$$

$$= \frac{5}{3}\cdot(33m + 0.8c\cdot 0) \qquad = \frac{5}{3}\cdot\left(0 + \frac{0.8c\cdot 33m}{c^2}\right)$$

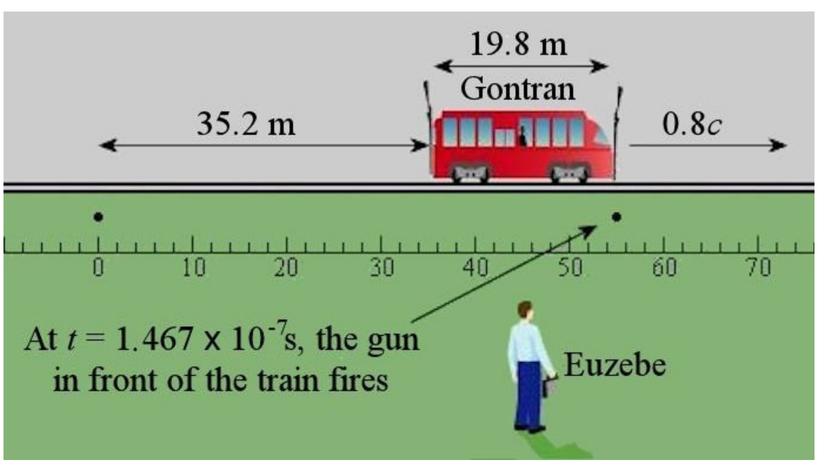
$$= 55m \qquad = 1.467\times 10^{-7}\,s$$

Thus, the distance between events, so between the bullet holes, is also 55 m according to Euzebe. There will be a hole at $x = 0$ and a hole at $x = 55$ m, as predicted by Gontran.

The results of the transformation also show the mistake made. For Euzebe, the guns do not fire at the same time! The time $t_2 - t_1 = 1.467 \times 10^{-7}$ s means that gun 2 (the one in front of the train) fires 1.467×10^{-7} s after the gun at the back of the train. During this time, the train, which is moving at 0.8*c,* has moved 35.2 m. Here is the solution of what happens according to Euzebe. First, the gun in the back fires and make a mark on the ground.



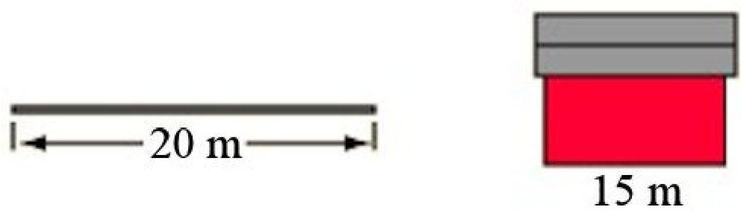
1.467×10^{-7} s later, the gun at the front of the train fires and leave a mark on the ground. Meanwhile, the train has moved forward by 35.2 m.



The position of the hole on the ruler is, therefore, 35.2 m + 19.8 m = 55 m according to Euzebe, as predicted by Gontran. There is no paradox since the predictions of both observers are the same.

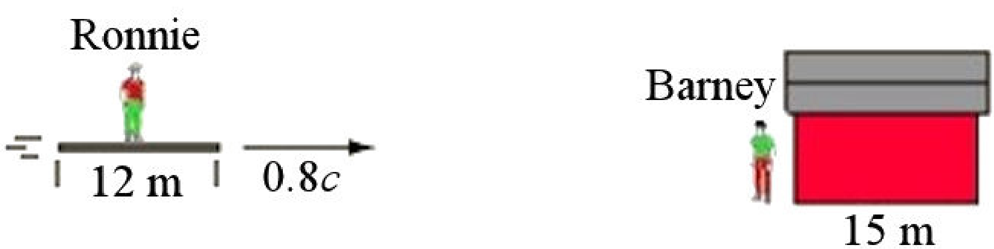
The Barn Paradox

Here is another situation that seems paradoxical when the solution is not done properly. In this example, there is a 15 m long barn and a 20 m long pole.



hyperphysics.phy-astr.gsu.edu/hbase/relativ/polebarn.html

The pole is too long to enter into the barn unless it goes very fast. Ronnie takes this pole and runs at 0.8*c*. Because of length contraction, the pole is only 12 m long then.



Barney, who is beside the door of the barn, opens the door to let the pole in and can close the door since the pole can fit entirely inside the barn.

This situation seems paradoxical, however, if Ronnie's point of view is taken. For him, the pole is at rest and the barn is coming towards him at 0.8*c*.

For Ronnie, the pole is 20 m long while the barn is only 9 m long because of length contraction. The pole cannot possibly fit into the barn and Barney cannot close the door!

So, can the door be shut or not? If it can be shut according to one observer, it must shut according to every observer. Relativity changes the time at which an event occurs, but events cannot appear or disappear with a change of reference frames.

To solve this mystery, it will be assumed first that the back wall of the barn shatters when it is hit by the pole. In this case, Barney can surely shut the door because the pole gets out on the other side of the barn. Notice, however, that the door shuts before the wall breaks according to Barney, while the wall breaks before the door shuts according to Ronnie. The order of the two events is reversed depending on the observer. This is quite possible in relativity since the time at which an event occurs changes from one observer to another. It is possible that this change reverses the order of the events. Don't worry, this inversion is impossible if an event is the cause of the other!

Let's now assume the wall doesn't shatter upon impact. In this case, the solution is more subtle. It will be shown a little further than nothing can travel faster than light according to the equations of relativity. This means that when the front of the pole hits the wall and stops, the back end of the pole does not stop immediately! Before the back end of the pole stops, the information saying to the back end that it has to stop must travel to the back end of the pole. As this information cannot go faster than light, the back of the pole continues to move forwards for a while before stopping. This motion of the back end of the pole that takes place while the front end is stopped makes the pole even shorter than the contracted length.

So, this is what is happening according to Barney. The pole enters the barn, and he shuts the door. Then, the front end of the pole hits the wall while the back end of the pole continues its motion until the length of the pole is only 6.67 m. Then, the information arrives at the back end of the pole saying that the pole is now at rest. But this pole is 20 m long at rest. So, it grows at the speed of light until it reaches 20 m long, and the door gets smashed.

In Ronnie's reference frame, the moving barn hits the right end of the pole at rest. But the left end of the pole doesn't immediately know that there was a contact and stays at rest. While the right end of the pole is moving towards the left at $0.8c$, the left end stays still, and the pole gets shorter. This shortening of the pole continues until the information of the collision with the barn arrives at the left end of the pole. When the information arrives, the pole is only 4 m long and is smaller than the 9 m long barn. Barney can then shut the door. As the perch is 12 m long when it is moving at $0.8c$, it extends at the speed of light until it is 12 m long. The pole then smashes the door, because it is longer than the barn!

There is no paradox. Barney can close the door according to both observers.

Rest your brain a few moments while new connections between neurons are formed…

9.13 MOMENTUM AND ENERGY

Newton's laws were in agreement with Galilean relativity. Changing from one frame to another using Galilean transformations, the values of the momentum and the kinetic energy according to another observer are obtained. With these transformations, the momentum and the kinetic energy are conserved for every observer in an elastic collision.

With the changes made to relativity by Einstein, the results obtained with $p = mu$ and $E_k = ½mu^2$ are no longer valid (u is used since this is the symbol used for the speed of an object in this chapter). With these formulas, it can be shown that the momentum and the kinetic energy are not conserved anymore for every observer in an elastic collision when Lorentz transformations are used to pass from one frame to another. New formulas for the momentum and the kinetic energy have to be found to preserve the principles of conservation of p and E_k.

Momentum

To find the new formula of the momentum, a collision where the momentum is obviously conserved for an observer is considered. Then, this collision is examined in the referential of another observer to determine how the momentum formula must be changed so that there is also momentum conservation for this observer.

Here is this proof.
http://physique.merici.ca/waves/proofprel.pdf

Then, the following formula for momentum of an object of mass m moving at speed u is obtained.

Momentum

$$p = \frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma mu$$

(γ is calculated with u, the speed of the object.)

Kinetic Energy

Kinetic energy can be found on the assumption that the work is equal to the variation of kinetic energy. Starting from zero velocity, the change in kinetic energy is the final kinetic energy. By accelerating in the direction of the x-axis, we have

$$E_k = W$$

$$= \int F dx$$

$$= \int \frac{dp}{dt} dx$$

$$= \int \frac{dx}{dt} dp$$

$$= \int u dp$$

With an integration by part, we arrive at

$$E_k = pu - \int p du$$

If the object is accelerated from zero velocity to a certain final velocity *u*, we have

$$\begin{aligned}
E_k &= \left[pu \right]_0^u - \int_0^u p du \\
&= \left[\frac{mu}{\sqrt{1-\frac{u^2}{c^2}}} u \right]_0^u - \int_0^u \frac{mu}{\sqrt{1-\frac{u^2}{c^2}}} du \\
&= \left[\frac{mu}{\sqrt{1-\frac{u^2}{c^2}}} u \right]_0^u + m \left[\frac{\left(c^2 - u^2\right)}{\sqrt{1-\frac{u^2}{c^2}}} \right]_0^u \\
&= \frac{mu^2}{\sqrt{1-\frac{u^2}{c^2}}} + m \frac{\left(c^2 - u^2\right)}{\sqrt{1-\frac{u^2}{c^2}}} - mc^2 \\
&= \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} - mc^2
\end{aligned}$$

Thus

Kinetic Energy

$$E_k = \left(\gamma - 1\right) mc^2$$

(γ is calculated with *u,* the speed of the object.)

This result is very different from Newton's physics ½mu^2. However, at low speeds, we have

$$\left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1 \right) mc^2 \approx \left(1 + \frac{u^2}{2c^2} + \ldots - 1 \right) mc^2 = \frac{1}{2} mu^2 + \ldots$$

The link is subtle, but, at low speed, the energy is the same as in Newton's physics.

Internal Energy and Mass

Relativity gives more than a new formula for kinetic energy. It also leads a link between mass and energy that applies to all forms of energy.

Energy and mass

$$E_{mass} = mc^2$$

(See how this equation is obtained at in the following document: https://physique.merici.ca/waves/proofErel.pdf)

The equation indicates that mass corresponds to a certain amount of energy and that energy corresponds to a certain amount of mass, regardless of the type of energy. This energy that corresponds to a certain mass is called *mass energy*.

This formula was first obtained (implicitly) by Poincaré in 1900, but it was Einstein who made it a general principle in 1905. (However, it was later realized that the all the proofs of $E = mc^2$ given by Einstein were incomplete. The first correct proof was made by Max von Laue in 1911, and an even more general proof was made by Felix Klein in 1918. It's a little bit ironic that the equation most often associated with Einstein wasn't first obtained by Einstein and wasn't proven correctly by Einstein…)

What Does $E_{mass} = mc^2$ Mean?

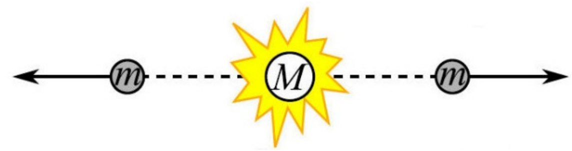
Contrary to what is often said, the equation does not mean that mass can be transformed into energy or energy into mass. That's not what this equation means at all. The equation says instead that mass and energy are equivalent. Let's see what that means.

In a closed system (no energy added or removed from the system), energy is conserved. Since the equation $E = mc^2$ says that mass and energy are equivalent, mass must also be conserved. Thus, the energy and the mass before a transformation are therefore always equal to the energy and mass after a transformation if the system is closed. However, you must make sure to count the total mass by adding the masses of the objects and the mass of all types of energy.

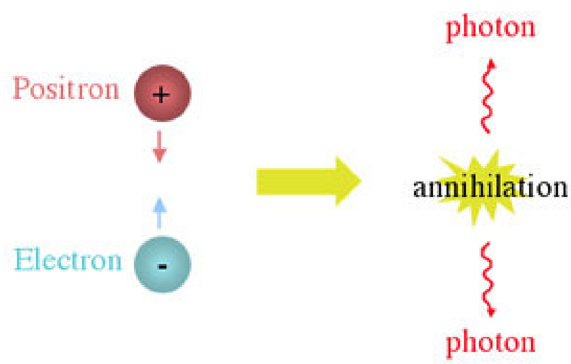
Suppose that two 2 kg balls (the mass are the mass of the objects when they are not moving) arrive with identical speeds to make a perfectly inelastic collision. After the collision, the two balls form a 4 kg object at rest, and the kinetic energy has been converted into heat. If we were to measure the mass of this 4 kg object really precisely, then we would find that its mass is greater than 4 kg because heat, which is a form of energy, has mass. It would therefore seem that the mass has increased in this situation. In reality, it has remained the same. Before the collision, the total mass is 2 kg + 2 kg + the mass of the kinetic energy of the balls. After the collision, the mass is 4 kg + the mass of thermal energy. Since thermal energy is equal to kinetic energy, the masses are identical before and after the collision.

We would only see the difference in mass before and after the collision if the mass of the 2 kg objects that will hit each other were measured before giving them kinetic energy.

Sometimes, it happens that a particle decays into 2 lighter particles (such as particles called *pions* that can decay into 2 photons for example). When this happens, it is easy to think that some of the mass has been transformed into kinetic energy. It is true that the mass of the particles after decay is smaller than the mass of the initial particle, but this does not mean that the total mass has changed. When the mass of the kinetic energy is added, the mass remains the same. Suppose that a particle of mass M decays into 2 particles of mass m. Before decay, the total mass is equal to the mass of the particle having mass M. After the decay, the total mass is equal to the sum of the mass of the 2 particles of mass m and the mass of the kinetic energy of these particles. This sum is exactly equal to M, which means that the total mass of the system has not changed.



Annihilation of matter and antimatter is often mentioned as a case of the transformation of mass into energy. When a particle of matter (an electron, for example) encounters a particle of antimatter of the same type (an antielectron, also called a positron or positron, in the case of the electron), the two particles disappear completely (they are said to *annihilate each other)* creating photons.



astronomy.swin.edu.au/cosmos/P/Positron

However, the mass did not turn into energy if this situation is examined carefully. The photons emitted have energy and therefore have mass. The total mass of these photons is equal to the mass of the two particles present before annihilation. The energy stayed the same and the mass did not change during annihilation.

If the system is not closed, the mass will increase if energy is added, or the mass will decrease if energy is taken away from the system. The change in mass is calculated with the change in energy with $E = mc^2$.

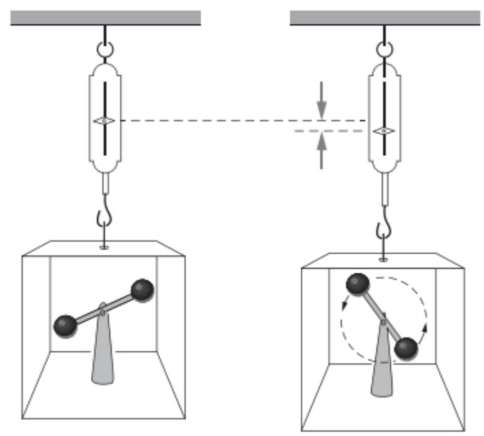
If a brick is lifted, energy is given to the brick-Earth system since the gravitational energy increases. So, this means that the mass of the Earth-brick system will increase a bit. For the same reason, the mass of the Earth-Moon system would increase if energy were given to the system to move the Moon away from Earth. The mass would increase even more if the Moon was given enough energy to leave the Earth. Thus, the Earth and the Moon together have a smaller mass than the mass they have when the Earth and the Moon are separated. The difference in mass is related to the difference in energy by $E = mc^2$.

In the same way, the mass of the hydrogen atom increases if energy is given to an electron in the orbitals. If enough energy is given to remove the electron, the mass will increase even more. This means that the mass of a separate proton and electron is greater than the mass of the hydrogen atom.

In most cases, the percentage of mass change is so small that it is impossible to measure. It has never been possible to measure the change in mass when a brick is lifted or when an electron is removed from an atom.

The only case where the change in mass is large enough to be measurable that we will see here is the atomic nuclei. To remove a proton from an atomic nucleus, energy must be given to the proton, and this added energy will make the mass increase. If all the protons and neutrons are removed from a nucleus one after the other, the mass increases even more. This means that the mass of the atomic nucleus is smaller than the mass of all the protons and neutrons when they are separated. In this case, the variation in mass of the order of 1% and this variation has been measured. We will examine this variation in mass in the chapter dealing with nuclear physics.

Now let's imagine that there is an object that can rotate inside a box. If the object starts to rotate, the total mass of the box will increase because the object now has kinetic energy and, as the kinetic energy has mass, the mass increases. The difference in mass is equal to the mass of the kinetic energy of the rotating object



physics.stackexchange.com/questions/411022/an-experiment-about-relativistic-effect-on-weight

There is a small subtlety, however. The mass of the box increases as long as the energy used to put the masses in rotation comes from outside the box (so, if this is an open system). This would be the case if the motor that set the masses in rotation is plugged into a power outlet outside the box.

However, there would be no difference in mass if the engine was powered by a battery that is inside the box (in which case, this is a closed system). In this case, the mass increases because the kinetic energy of the masses has increased, but the mass decreases because the battery has lost energy. Since the energy gained in kinetic energy is equal to the energy lost by the battery, there is no energy variation in the box and, therefore, no mass variation.

Calculating Energy From Mass Difference

Even if the mass remains the same in a closed system, it is sometimes possible to calculate the energy released or the energy absorbed during a transformation with the change in mass. This can be done by considering only the masses of objects without counting the mass of their kinetic energy. Here's an example of how to do it.

When an atomic bomb explodes, energy and mass are conserved. Some of the internal energy of the atomic nuclei is transformed into kinetic energy of the atoms after the explosion. The mass of the kinetic energy of all the fragments is exactly equal to the mass of the energy released by the atoms. However, if all the atoms from the bomb explosion were found long after the explosion, we would find that the mass is smaller than that of the bomb before the explosion. The mass is then smaller because fragments of the explosion will have lost their kinetic energy by then. Since the mass of the kinetic energy is no longer present, the total mass decreases by a value corresponding to the mass of this energy. Since the kinetic energy was equal to the energy released, the drop in mass is also equal to the mass of the energy released by the explosion. The difference in mass is often not very great. Indeed, the energy equivalent to one gram of matter

$$\begin{aligned} E_{mass} &= mc^2 \\ &= 0.001kg \cdot \left(3\times 10^8 \tfrac{m}{s}\right)^2 \\ &= 9\times 10^{13} J \end{aligned}$$

corresponds approximately to the energy released during the explosion of the atomic bomb in Hiroshima! Thus, all the fragments of this bomb after the explosion had, once they had lost their kinetic energy, a mass about 1 g smaller than the mass of the bomb before the explosion.

Relativistic Energy

The relativistic energy E of an object is defined as the sum of the mass energy of the object when it is at rest and the kinetic energy of the object.

Relativistic energy = Mass energy + Kinetic energy

Thus, we have

$$\begin{aligned} E &= mc^2 + E_k \\ &= mc^2 + (\gamma - 1)mc^2 \end{aligned}$$

(In these notes, the masses m or M are always the mass of objects when they are at rest.) Simplifying the equation then gives us.

Relativistic Energy

$$E = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} mc^2 = \gamma mc^2$$

(γ is calculated with u, the speed of the object.)

There is a variant to this equation: it's a formula that gives the total mass of a moving object. An object that is moving has kinetic energy, and this energy has mass. The total mass of the moving object is therefore greater than the mass of the object at rest. The total mass of the object is the sum of its mass at rest and the mass of its kinetic energy.

$$\begin{aligned} m_{tot} &= m + m_{Kinetic\ Energy} \\ &= m + \frac{E_k}{c^2} \\ &= m + \frac{(\gamma - 1) mc^2}{c^2} \\ &= m + (\gamma - 1) m \\ &= \gamma m \end{aligned}$$

This equation confirms that the mass of the moving object is greater than the mass of the same object at rest (since γ is always greater than 1). The mass is not greater because there is more matter, but because the mass of kinetic energy is added to the mass of the object at rest. This equation of total mass as a function of velocity is in fact the equation of relativistic energy $E = \gamma mc^2$ divided by c^2.

The equation of the total mass of a moving object is not very useful. What really matters when an object is in motion is its momentum or its energy.

The Electronvolt

A small note to complete this section. When the energy of a subatomic particle (such as a proton or an electron) is calculated, the values obtained are generally very small if calculated in joules. Physicists usually use the electronvolt to measure the energy of such small particles. There is an exact definition of the electronvolt (that we will see in electricity and magnetism) but we can be content here with this simple following definition.

Electronvolt (*eV*)

$$1eV = 1.602 \times 10^{-19} J$$

Example 9.13.1

What is the mass energy of an electron, whose mass is 9.11×10^{-31} kg?

The mass energy is

$$\begin{aligned} E_{mass} &= mc^2 \\ &= 9.11\times10^{-31}kg\cdot\left(3\times10^8\tfrac{m}{s}\right)^2 \\ &= 8.18\times10^{-14}J \\ &= 511keV \end{aligned}$$

Example 9.13.2

What is the kinetic energy of an electron travelling at 0.8c?

The kinetic energy is

$$\begin{aligned} E_k &= (\gamma-1)mc^2 \\ &= \left(\frac{1}{\sqrt{1-\frac{(0,8c)^2}{c^2}}}-1\right)mc^2 \\ &= \left(\frac{5}{3}-1\right)mc^2 \\ &= \frac{2}{3}mc^2 \end{aligned}$$

As we calculated in the previous example that mc^2 = 511 keV for an electron, we arrive at

$$\begin{aligned} E_k &= \left(\frac{2}{3}\right)\cdot 511keV \\ &= 340keV \end{aligned}$$

Note that with ½mu^2, the kinetic energy would have been 164 keV.

Example 9.13.3

How much energy must be given to a 100-ton spaceship initially at rest so that it moves at 0.8c?

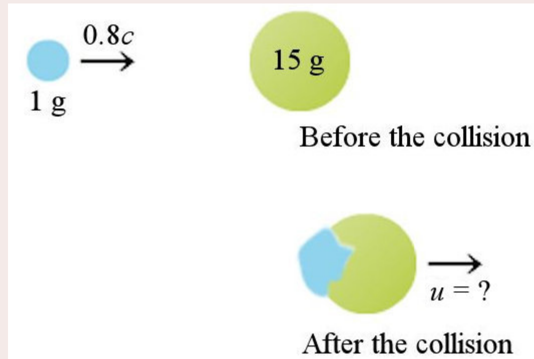
The energy is

$$\begin{aligned} \Delta E &= E_f - E_i \\ &= \gamma_f mc^2 - \gamma_i mc^2 \\ &= (\gamma_f - \gamma_i)mc^2 \\ &= \left(\frac{5}{3}-1\right)\cdot 100\,000kg\cdot\left(3\times10^8\tfrac{m}{s}\right)^2 \\ &= 6\times10^{21}J \end{aligned}$$

As all the power stations on Earth produce approximately 6 x 10^{20} J each year (2021 diagram), the energy that must be given to the spaceship corresponds to 10 years of all the electricity produced on Earth! Such a feat will not be achieved soon! It looks so easy in science fiction movies…

Example 9.13.4

A 1-gram object moving at 0.8*c* makes a perfectly inelastic collision in one dimension with a 15 g object at rest.



physics.tutorvista.com/momentum/inelastic-collision.html

a) What is the speed of the 16 g object after the collision?

As in any collision, momentum is conserved. Before the collision, the momentum is

$$\begin{aligned} p_{before} &= \gamma m u \\ &= \frac{1}{\sqrt{1-(0.8)^2}} \cdot 0.001 kg \cdot 0.8c \\ &= 400\,000 \tfrac{kgm}{s} \end{aligned}$$

After the collision, the momentum is

$$\begin{aligned} p_{after} &= \gamma m u \\ &= \gamma \cdot 0.016 kg \cdot u \end{aligned}$$

Since momentum is conserved, we have

$$\begin{aligned} p_{before} &= p_{after} \\ 400\,000 \tfrac{kgm}{s} &= \gamma \cdot 0.016 kg \cdot u \\ 2.5\times10^7 \tfrac{m}{s} &= \gamma u \end{aligned}$$

It only remains to solve this equation for *u*.

$$\begin{aligned} 2.5\times10^7 \tfrac{m}{s} &= \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}} u \\ 2.5\times10^7 \tfrac{m}{s} \cdot \sqrt{1-\left(\tfrac{u}{c}\right)^2} &= u \\ \left(2.5\times10^7 \tfrac{m}{s}\right)^2 \cdot \left(1-\left(\tfrac{u}{c}\right)^2\right) &= u^2 \end{aligned}$$

$$\left(2.5\times10^{7}\tfrac{m}{s}\right)^{2}-\frac{\left(2.5\times10^{7}\tfrac{m}{s}\right)^{2}}{c^{2}}u^{2}=u^{2}$$

$$\left(2.5\times10^{7}\tfrac{m}{s}\right)^{2}=u^{2}+\frac{\left(2.5\times10^{7}\tfrac{m}{s}\right)^{2}}{c^{2}}u^{2}$$

$$\left(2.5\times10^{7}\tfrac{m}{s}\right)^{2}=\left(1+\frac{\left(2.5\times10^{7}\tfrac{m}{s}\right)^{2}}{c^{2}}\right)u^{2}$$

$$\left(2.5\times10^{7}\tfrac{m}{s}\right)^{2}=\tfrac{145}{144}u^{2}$$

$$u=\sqrt{\tfrac{144}{145}}\cdot2.5\times10^{7}\tfrac{m}{s}$$

$$u=2.4914\times10^{7}\tfrac{m}{s}$$

$$u=0.08305c$$

(With Newton's laws, the answer would have been $u = 0.05c$.)

b) How much kinetic energy is lost in the collision?

Before the collision, the kinetic energy is

$$\begin{aligned}E_{k\ before}&=(\gamma-1)mc^{2}\\&=\left(\frac{1}{\sqrt{1-(0.8)^{2}}}-1\right)\cdot0.001kg\cdot\left(3\times10^{8}\tfrac{m}{s}\right)^{2}\\&=6\times10^{13}J\end{aligned}$$

After the collision, the kinetic energy is

$$\begin{aligned}E_{k\ after}&=(\gamma-1)mc^{2}\\&=\left(\frac{1}{\sqrt{1-(0.08305)^{2}}}-1\right)\cdot0.016kg\cdot\left(3\times10^{8}\tfrac{m}{s}\right)^{2}\\&=4.9913\times10^{12}J\end{aligned}$$

Thus, the change in kinetic energy is

$$\begin{aligned}\Delta E_{k}&=4.9913\times10^{12}J-6\times10^{13}J\\&=-5.5\times10^{13}J\end{aligned}$$

The kinetic energy has, therefore, decreased by 5.5×10^{13} J in this collision. (This energy would normally be in the form of heat after an inelastic collision but the energy here is practically equal to the energy released by the atomic bomb of Hiroshima. The 16 g object would surely be vaporized.)

Relation Between *E* and *p*

There is an interesting link between *E* and *p*. Let's calculate what is $E^2 - (pc)^2$.

$$\begin{aligned} E^2 - (pc)^2 &= \gamma^2 m^2 c^4 - \gamma^2 m^2 u^2 c^2 \\ &= \gamma^2 \left(c^2 - u^2\right) m^2 c^2 \\ &= \gamma^2 \left(1 - \tfrac{u^2}{c^2}\right) m^2 c^4 \\ &= \frac{1}{1 - \frac{u^2}{c^2}} \left(1 - \tfrac{u^2}{c^2}\right) m^2 c^4 \\ &= m^2 c^4 \end{aligned}$$

Thus

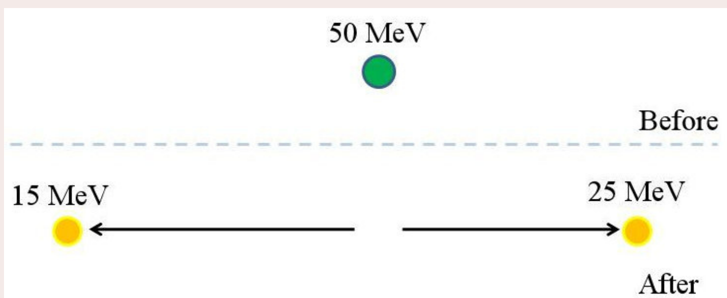
Link Between Relativistic Energy and Momentum

$$E^2 - (pc)^2 = \left(mc^2\right)^2$$

Note that the term on the right is a relativistic invariant. All the observers get the same value when they calculate $E^2 - (pc)^2$.

Example 9.13.5

A particle at rest having a 50 MeV mass energy decays into two particles which move in opposite directions. The mass energy of one of these particles is 15 MeV, and the mass energy of the other particle is 25 MeV. What are the speeds of the two particles?



As in any process, the relativistic energy is conserved. Since the mass energy is 50 MeV before the decay and 40 MeV after the decay, the kinetic energy of the two particles after the decay must be 10 MeV.

Since the momentum is zero before the decay, it must also be zero after the decay. So, we have the following equations.

$$E_{k1} + E_{k2} = 10MeV$$
$$p_1 + p_2 = 0$$

Using the formulas for *p* and E_k, these equations become

$$\left(\frac{1}{\sqrt{1-\left(\frac{u_1}{c}\right)^2}}-1\right)\cdot 15MeV+\left(\frac{1}{\sqrt{1-\left(\frac{u_2}{c}\right)^2}}-1\right)\cdot 25MeV=10MeV$$

$$\left(\frac{1}{\sqrt{1-\left(\frac{u_1}{c}\right)^2}}\right)\cdot\frac{15MeV}{c^2}u_1+\left(\frac{1}{\sqrt{1-\left(\frac{u_2}{c}\right)^2}}\right)\cdot\frac{25MeV}{c^2}u_2=0$$

We then have two equations with two unknowns, which can be solved.

But as it is very difficult to solve this system, another approach will be used. This approach uses the following equation.

$$E^2-(pc)^2=(mc^2)^2$$

Thus, since the momentum is conserved, we have

$$\begin{aligned}p_1&=-p_2\\ p_1^2&=p_2^2\\ p_1^2c^2&=p_2^2c^2\\ E_1^2-m_1^2c^4&=E_2^2-m_2^2c^4\end{aligned}$$

Since

$$E_1+E_2=50MeV$$

the equation becomes

$$\begin{aligned}E_1^2-m_1^2c^4&=(50MeV-E_1)^2-m_2^2c^4\\ E_1^2-m_1^2c^4&=(50MeV)^2-100MeV\cdot E_1+E_1^2-m_2^2c^4\\ -m_1^2c^4&=(50MeV)^2-100MeV\cdot E_1-m_2^2c^4\end{aligned}$$

Solving for the energy of particle 1, the result is

$$E_1=\frac{(50MeV)^2-m_2^2c^4+m_1^2c^4}{100MeV}$$

Using the values for the mass energy, the energy is

$$\begin{aligned}E_1&=\frac{(50MeV)^2-(25MeV)^2+(15MeV)^2}{100MeV}\\ &=21MeV\end{aligned}$$

From there, it can easily be found that

$$E_2 = 50MeV - E_1$$
$$= 29MeV$$

As the relativistic energy is the sum of the mass energy and the kinetic energy, it can easily be found that the kinetic energy is 6 MeV for particle 1 and 4 MeV for particle 2. The sum of these kinetic energies is indeed 10 MeV, as predicted.

The value of γ for each particle can then be found with the relativistic energy formula.

$$E_1 = \gamma_1 m_1 c^2 \qquad E_2 = \gamma_2 m_2 c^2$$
$$21MeV = \gamma_1 \cdot 15MeV \qquad 29MeV = \gamma_1 \cdot 25MeV$$
$$\gamma_1 = 1.4 \qquad \gamma_1 = 1.16$$

With the γ's, the speeds can be found.

$$1.4 = \frac{1}{\sqrt{1-\left(\frac{u_1}{c}\right)^2}} \qquad 1.16 = \frac{1}{\sqrt{1-\left(\frac{u_2}{c}\right)^2}}$$
$$u_1 = 0.69985c \qquad u_2 = 0.50679c$$

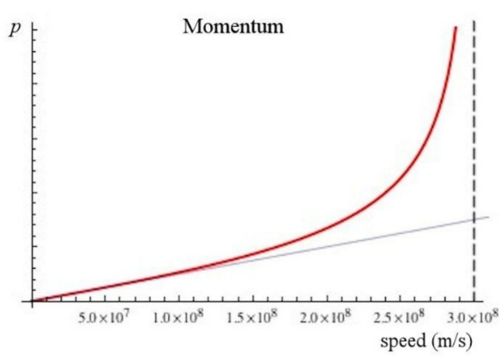
If E Is Conserved, Then p Must Also Be Conserved

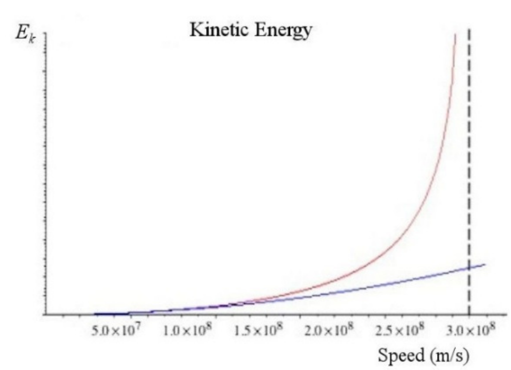
Here's a small interesting note. If the relativistic energy is conserved in a collision, then it can be proven, using the principle of relativity, that the momentum is automatically a conserved quantity.

The proof can be seen here: http://physique.merici.ca/waves/ConservationEp2-eng.pdf

Maximum Speed

Let's have a look at the graphs of the momentum and of the kinetic energy of an object as functions of its speed. On each graph, the values according to Newton's Physics (in blue) and the values according to Einstein's physics (in red) are plotted.





Three things can be noted.

1) The values given by the relativistic formulas are always larger than those given by the non-relativistic formulas. An example of this can be seen in example 9.13.1 (340 keV vs. 164 keV).

2) The values are virtually identical for small speeds (say, less than 20% of the speed of light). The non-relativistic equations are, therefore, very good approximations at low speeds.

3) The values of p and E_k are infinite at $u = c$.

This last point shows that it is impossible for an object to reach the speed of light because it will take an infinite amount of energy for the object to reach this speed. The speed of an object cannot exceed the speed of light.

Maximum Speed

No object with a non-vanishing mass can reach the speed of light.

The media don't seem to be aware of this fact.

www.epicfail.com/2012/10/16/news-fail-5/

However, this speed can be reached if the mass is zero. In this case, the energy, which is

$$E = \gamma mc^2$$

may not be infinite even if γ is infinite because $m = 0$. But then, the equation gives $E = 0$ if γ is not infinite. If the energy is zero, then this object does not exist. Therefore, an object without any mass can only exist if its speed is equal to the speed of light.

The only thing known that has no mass in all current theories is the photon, which is a light particle (we will talk about it in the next chapter). Therefore, for photons, we have

Photon Speed

Photons always travel at the speed of light because their mass is zero.

Moreover, for photons, the relation between E and p is (we will E_γ and p_γ for the energy and momentum of the photon)

$$E_\gamma^2 - \left(p_\gamma c\right)^2 = \left(mc^2\right)^2$$

$$E_\gamma^2 - \left(p_\gamma c\right)^2 = 0$$

since m is 0. Therefore,

Link Between Momentum and Energy for Photons

$$E_\gamma = p_\gamma c$$

Faster Than *c*

The fact that massive objects cannot exceed the speed of light does not mean that nothing can go faster than the speed of light. Some "things" can go faster than light, but they are not material things.

For example, the bright spot made by a laser pointer on a screen can go faster than light. By turning the pointer by 5°, the spot moves on the screen. If the screen is far, the displacement of the spot can be very large. If the screen were 1 billion km away and the pointer is turned by 5° in 1 second, the spot would move 87 000 000 km in 1 second on the screen. This speed is greater than the speed of light. However, no particle has made this displacement since the spot is constantly made of new photons.

SUMMARY OF EQUATIONS

γ Factor

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Time Dilation

$$\Delta t = \gamma \Delta t_0 = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Length Contraction

$$L = L_0\sqrt{1-\frac{v^2}{c^2}} = \frac{L_0}{\gamma}$$

Approximations of the γ Factor if $v << c$

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1+\frac{v^2}{2c^2}+\ldots$$

$$\sqrt{1-\frac{v^2}{c^2}} \approx 1-\frac{v^2}{2c^2}+\ldots$$

Relativistic Doppler Effect

$$T' = T_0\sqrt{\frac{c-v}{c+v}}$$

$$f' = f_0\sqrt{\frac{c+v}{c-v}}$$

(v is the speed of the source according to a motionless observer.)
The sign convention for speed stays the same as it was. The positive direction is from the source towards the observer. The speed of the source is positive if it is heading towards the observer and negative if it moves away from the observer.

Lorentz Transformations (1)

$$x' = \gamma(x - vt) \qquad x = \gamma(x' + vt')$$

$$y' = y \qquad y = y'$$

$$z' = z \qquad z = z'$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \qquad t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

(v is the speed of an observer according to another observer.)

Lorentz Transformations (2)

$$\Delta x' = \gamma(\Delta x - v\Delta t) \qquad \Delta x = \gamma(\Delta x' + v\Delta t')$$

$$\Delta y' = \Delta y \qquad \Delta y = \Delta y'$$

$$\Delta z' = \Delta z \qquad \Delta z = \Delta z'$$

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \qquad \Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)$$

(v is the speed of an observer according to another observer.)

The Interval

$$I = c^2 (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$
$$I' = I$$

Proper Time Between Two Events

$$\Delta t_0 = \tfrac{1}{c}\sqrt{I}$$

Proper Distance Between Two Events

$$\Delta \sigma = \sqrt{-I}$$

Velocities Transformations

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \qquad u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}$$

$$u'_y = \frac{u_y\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{vu_x}{c^2}} \qquad u_y = \frac{u'_y\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{vu'_x}{c^2}}$$

$$u'_z = \frac{u_z\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{vu_x}{c^2}} \qquad u_z = \frac{u'_z\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{vu'_x}{c^2}}$$

(v is the speed of an observer according to another observer.)

Momentum

$$p = \frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma mu$$

Kinetic Energy

$$E_k = (\gamma - 1) mc^2$$

(γ is calculated with u, the speed of the object.)

Energy and Mass

$$E_{mass} = mc^2$$

Relativistic Energy

Relativistic Energy = Internal Energy + Kinetic Energy

$$E = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} mc^2 = \gamma mc^2$$

Link Between Relativistic Energy and Momentum

$$E^2 - (pc)^2 = (mc^2)^2$$

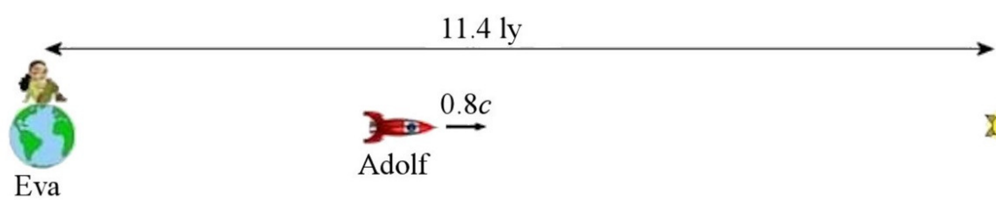
Link Between Momentum and Energy for Photons

$$E_\gamma = p_\gamma c$$

EXERCISES

9.7 Time Dilation

1. Adolf leaves Earth at 80% of the speed of light to go to the star Procyon, 11.4 ly away from Earth.



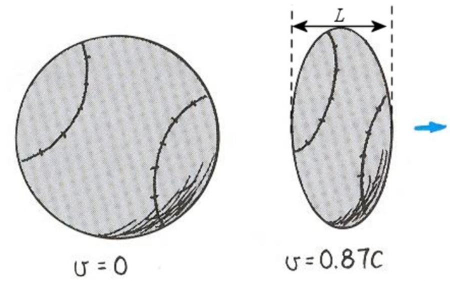
a) What is the duration of the trip according to Eva, who has remained on Earth?
b) What is the duration of the trip according to Adolf?

2. A clock remains on Earth while another clock is moving relative to the Earth at a speed of 0.6c. Initially, the two clocks are synchronized. The moving clock goes 90 million km away (according to observers on Earth) and comes back on Earth at the same speed. What will the difference between the time indicated by the moving clock and the clock which has remained on Earth be once the moving clock has returned to Earth?

3. Two twins, Octavius and Augustus, leave Earth at the same time, the day of their 20th birthday, to go on another planet located 12 ly from Earth. Octavius travels at 0.8c, and Augustus travels at 0.6c. What will the age of the twins be when Augustus finally arrives on the planet?

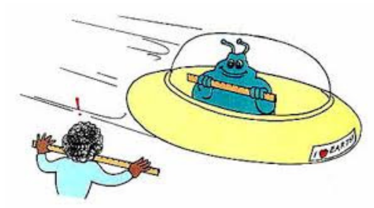
9.8 Length Contraction

4. A baseball has a 7.2 cm diameter at rest. What will the width of the ball (L in the diagram) be if it moves at 87% of the speed of light?



www.mattshields.com/?cat=14&paged=2

5. An alien passes beside Tom. Tom and the alien are both holding a 1 m long ruler (length at rest) as shown in the diagram. According to Tom, the ruler held by the alien is only 80 cm long.



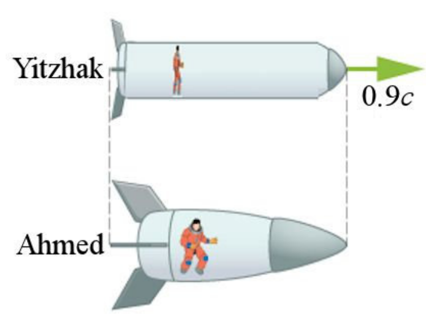
www.quarkology.com/12-physics/92-space/92E-relativity.html

a) What is the speed of the alien according to Tom?
b) What is the length of Tom's ruler according to the alien?

6. The distance between the Earth and a star is 500 ly according to an observer on Earth. How fast must a spaceship travel so that the distance between the Earth and the star is only 20 ly according to the observers in the spaceship?

7. Raoul wants to travel to the star Betelgeuse, located 643 ly away from the Earth (according to an observer on Earth). How fast must he go so that the trip lasts only 20 years according to Raoul?

8. Ahmed looks at Yitzhak's spaceship passing at $0.9c$. Ahmed noted that, according to him, the two spaceships have the same length. What is the ratio of the lengths of the spaceships according to Yitzhak?

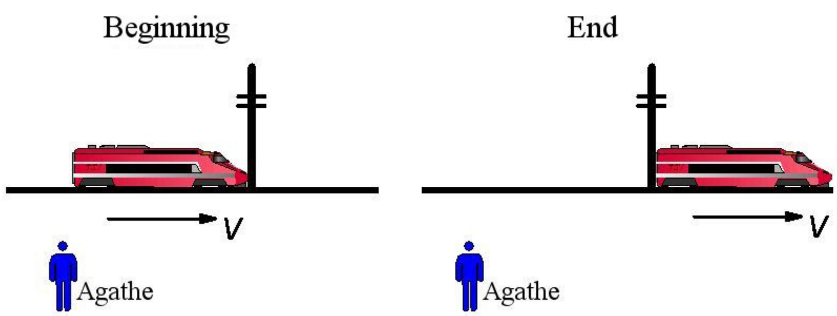


s3-us-west-2.amazonaws.com/oa2/docfiles/
54662e916f7065735bdd0100/54662e916f7065735bdd0100.html

9. A muon lives only 2.2 μs when it is at rest. Assume that newly created muons are launched at one end of a tunnel and are received at the other end of the tunnel. The tunnel is 1 km long.

a) How fast must a muon go so that it can travel 1 km during his lifetime?
b) At this speed, what is the length of the tunnel according to the muon?

10. Agathe measures the time it takes for a train going at 80% of the speed of light to pass beside a post. She starts her stopwatch when the front of the train is beside the pole, and she stops it when the back of the train is beside the pole.



renshaw.teleinc.com/papers/simiee2/simiee2.stm

On the train, Justin also measures the time it takes the train to past the post exactly as Agathe does it. The train is 2000 m long when it is at rest.

a) What is the length of the train according to Agathe?
b) How long does the train take to past the post according to Agathe?
c) How long does the train take to past the post according to Justin?

9.9 Doppler Effect with Light (Take 2)

11. A source on Earth emits a 98.1 MHz electromagnetic wave. What is the frequency received by Terence if he is in a spaceship moving at 95% of the speed of light,…

a) towards Earth?
b) away from Earth?

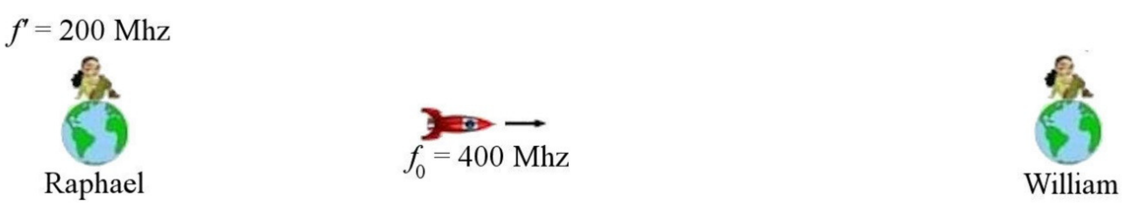
12. A star emits a light wave with a 550 nm wavelength (green). What is the wavelength of the wave received by an observer who sees the star moving away from him at 30% of the speed of light?

13. How fast must an observer move towards a red t-shirt (650 nm wavelength) so that the t-shirt looks blue to him (470 nm wavelength)?

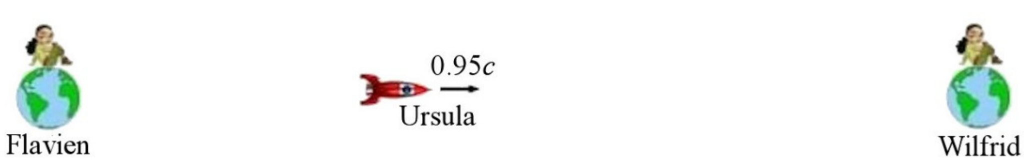
www.pinterest.com/pin/30610472438270660/

14. A source on Earth emits a 100 GHz electromagnetic wave towards a spaceship heading towards the Earth at 60% of the speed of light. The wave reflects off the spaceship and returns to Earth. Then, what is the frequency of the reflected waves received by the observers on Earth?

15. In the following situation, what is the frequency received by William if Raphael receives a 200 MHz frequency and the spaceship emits a 400 MHz frequency?

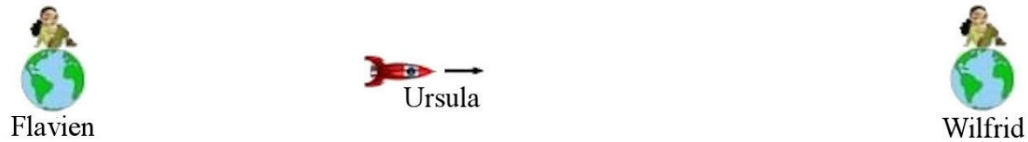


16. In the following situation, Ursula's spaceship emits a flash of light every 10 seconds (time according to Ursula).



a) What is the time between flashes according to Wilfrid's observations?
b) What is the time between flashes as seen by Wilfrid?
c) What is the time between flashes according to Flavien's observations?
d) What is the time between flashes as seen by Flavien?

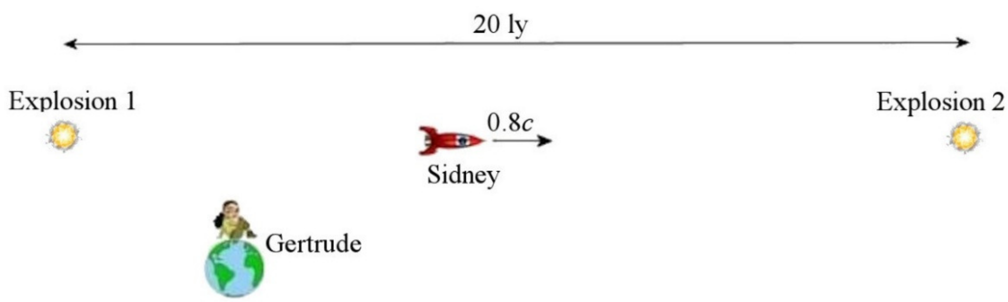
17. In the following situation, Ursula's ship emits flashes of light at regular intervals. Wilfrid sees a flash every 4 seconds, and Flavien sees a flash every 9 seconds.



a) What is the time between the flashes according to Ursula?
b) What is the speed of Ursula's ship according to Flavian and Wilfrid?

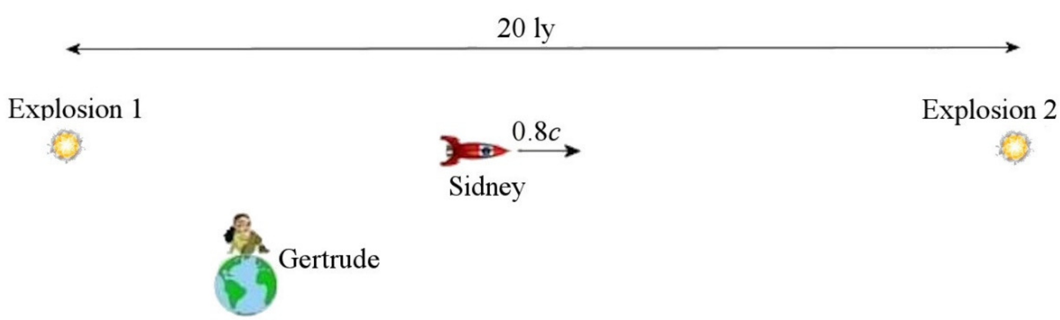
9.10 Lorentz Transformations

18. According to Gertrude, the two explosions are simultaneous and 20 ly from each other.



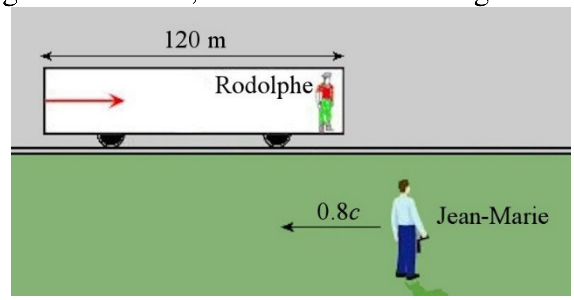
a) What is the time between the explosions according to Sidney? (Specify which explosion occurred first according to Sidney.)
b) What is the distance between the positions where the explosions occurred according to Sidney?

19. According to Gertrude, explosion 2 occurred 2 years before explosion 1, and the distance between the two positions where the explosions occurred is 20 ly.



a) What is the time between the explosions according to Sidney? (Specify which explosion occurred first according to Sidney.)
b) What is the distance between the positions where the explosions occurred according to Sidney?

20. A ray of light passes from the back to the front of a train. This train is moving at $0.8c$ relative to the ground. On the edge of the track, Jean-Marie is looking at the moving train. The following diagram illustrates the situation from the point of view of a passenger on the train (called Rodolphe). How long does it take for the light to pass from the back to the front of the train according to Jean-Marie?



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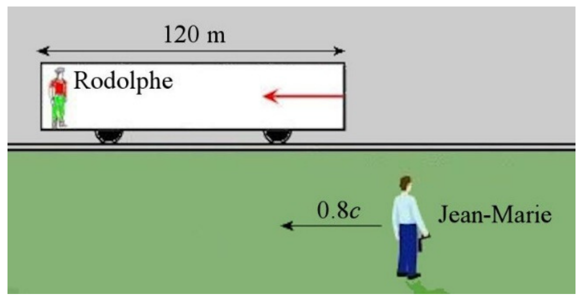
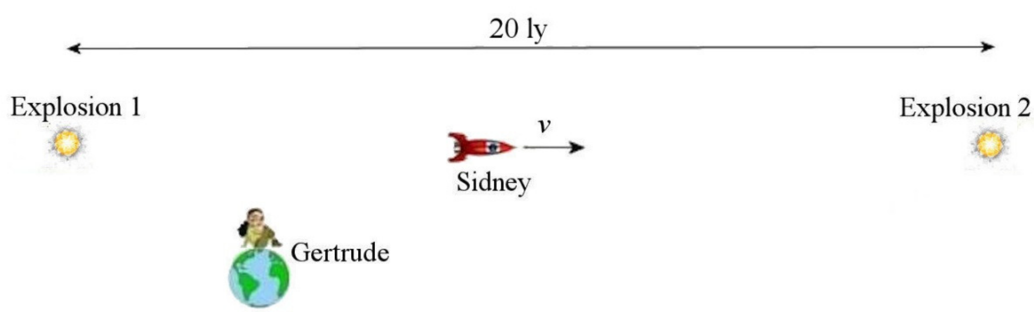
21. A ray of light passes from the front to the back of a train. This train is moving at $0.8c$ relative to the ground. On the edge of the track, Jean-Marie is looking at the moving train. The following diagram illustrates the situation from the point of view of a passenger on the train (called Rodolphe). How long does it take for the light to pass from the front to the back of the train according to Jean-Marie?



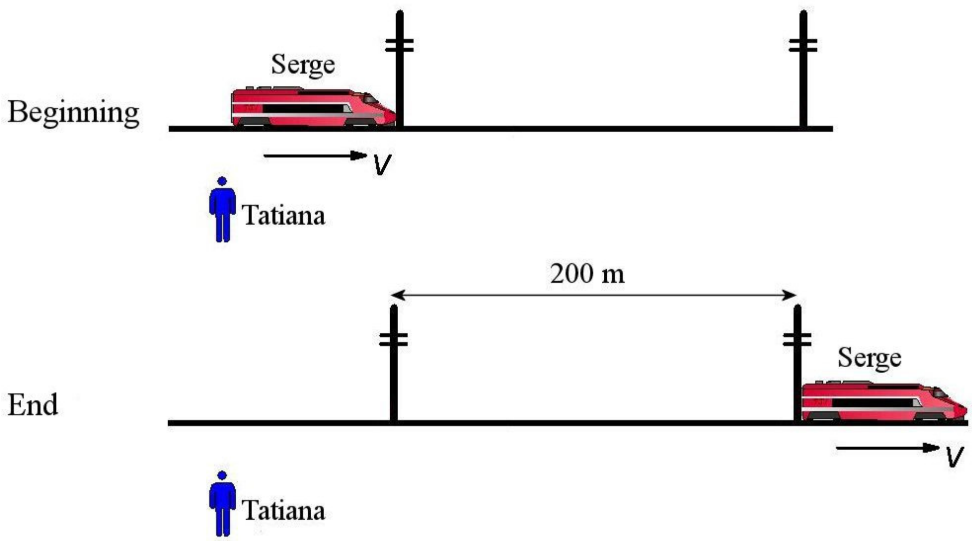
22. In the situation shown in the diagram, Gertrude measures that explosion 2 occurred 4 years before explosion 1.



a) What should the speed of Sidney be so that the explosions are simultaneous for him? (Even if the spaceship is drawn as a spaceship moving towards the right, it may have to go towards the left to observe simultaneous explosions.)

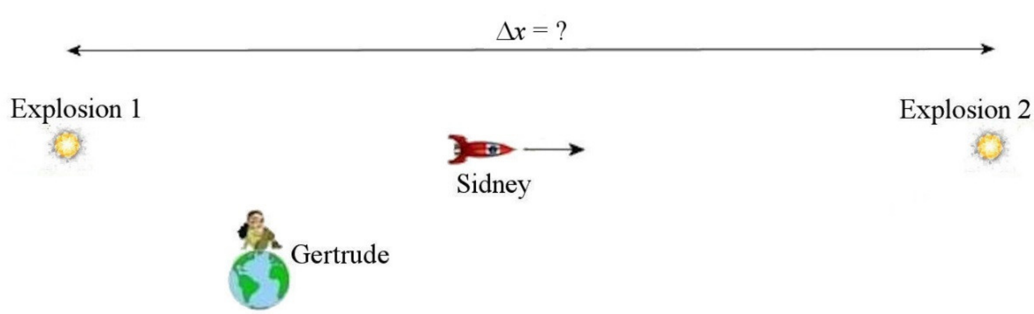
b) What is the distance between the events according to Sydney? (This distance is the proper distance since it is the distance between the events for an observer who observes that the events are simultaneous.)

23. Serge and Tatiana both measure the time it takes for a train going at 60% of the speed of light to go from one post to another. The distance between the poles is 200 m according to Tatiana. The train is 70 m long according to Tatiana. They start their stopwatch when the front of the train is beside the first pole, and they stop them when the back of the train is beside the second post.



a) What is the time measured by Tatiana?
b) What is the time measured by Serge?
c) What is the distance between the posts according to Serge?

24. In the situation shown in the diagram, Sidney observes that explosion 2 occurred 8 years before explosion 1 and that the distance between the positions where the explosions occurred is 5 ly. (The direction of the velocity shown in the diagram is correct.)



a) What is the distance between the positions where the explosions occurred according to Gertrude if explosion 2 occurred 7 years before explosion 1 according to Gertrude?
b) What is the speed of Sydney according to Gertrude?
c) What is the proper time between the events?

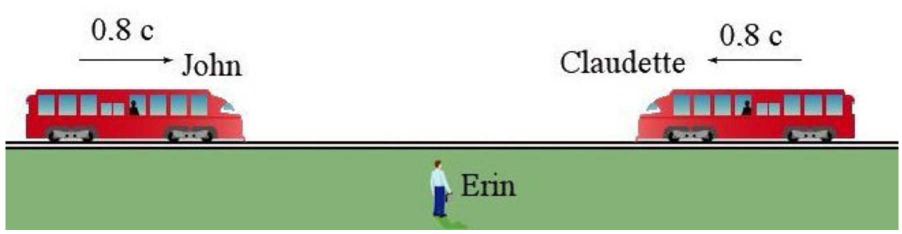
9.11 Velocities Transformations

25. Aline observes Kim's and Mike's spaceship heading towards each other with a speed of 0.95*c*. What is the speed of Mike's spaceship according to Kim?

26. According to Bertha, Oscar is travelling at 0.8*c* and is following Ivan who is travelling at 0.9*c*. Oscar launches a missile towards Ivan. According to Oscar, the speed of the missile is 0.25*c*.

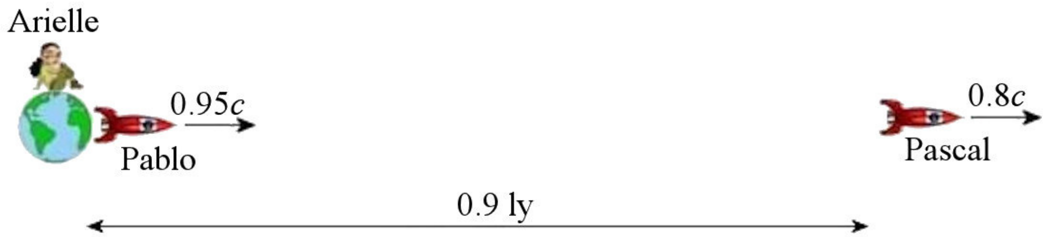
a) What is the speed of the missile according to Bertha?
b) Will the missile catch Ivan?

27. Two 246 m long trains (according to Erin) are heading towards each other as shown in the diagram. What is the length of John's train according to Claudette?



28. Two spaceships are heading towards each other as shown in the diagram. Annabelle's spaceship emits a 100 MHz signal (according to Annabelle). What is the frequency received by Esteban?

29. Pascal leaves Earth at 80% of the speed of light. When he is 0.9 ly away from Earth (distance according to Arielle, an observer on Earth), Pablo starts to pursue him at 95% of the speed of light.



a) What is Pablo's speed according to Pascal?
b) Who measures the proper time between the instant Pablo leaves the Earth and the instant Pablo catches up with Pascal?

How long will it take for Pablo to catch up with Pascal…

c) according to Arielle?
d) according to Pablo?
e) according to Pascal?

30. Paul is travelling towards a planet at 40% of the speed of light according to Lou, a female inhabitant of the planet. As Paul is not welcome there, Lou launches a missile at 80% of the speed of light to destroy Paul's spaceship. When the missile leaves the planet, Paul is 12 ly away from the planet according to Lou.

a) How much time elapses between the launch of the missile and the explosion of Paul's spaceship according to Lou?
b) What is the distance between the planet and Paul's spaceship according to Lou when the missile explodes?
c) What is the speed of the missile according to Paul?
d) How much time elapses between the launch of the missile and the explosion of Paul's spaceship according to Paul?
e) What is the distance between the planet and Paul's spaceship according to Paul when the missile explodes?

9.13 Momentum and Energy

31. The mass of a baseball is 145 g at rest.

a) What is the speed of the baseball if its kinetic energy is equal to the energy released by the explosion of Hiroshima's atomic bomb (6.3×10^{13} J)?
b) How many times must the energy of Hiroshima's atomic bomb be given to the baseball so that it reaches 95% of the speed of light?

32. A proton ($m = 1.673 \times 10^{-27}$ kg) is moving at 95% of the speed of light.

a) What is the momentum of this proton?
b) What is the kinetic energy of this proton (in MeV)?
c) What is the relativistic energy of this proton (in MeV)?
d) What is the mass of the proton at this speed?

33. How much energy (in keV) must be given to an electron ($m = 9.11 \times 10^{-31}$ kg) so that its speed changes from…

a) $0.7c$ to $0.8c$?
b) $0.8c$ to $0.9c$?

34. The Sun emits 3.83×10^{26} joules every second. If the Sun loses this energy every second, by how much does the mass of the Sun decreases every second?

35. A proton ($m = 1.673 \times 10^{-27}$ kg) has a 50 GeV relativistic energy. What is its momentum?

36. An electron ($m = 9.11 \times 10^{-31}$ kg) passes in a tunnel whose length is 10 km according to the observers on the surface of the Earth. The kinetic energy of the electron is 1 TeV (which is 10^{12} eV), always according to the observers on Earth.

a) What is the length of the tunnel in the reference frame of the electron?
b) How much time does it take for the electron to pass from one end of the tunnel to the other end according to the observers on the surface of the Earth?
c) How much time does it take for the electron to pass from one end of the tunnel to the other end in the reference frame of the electron?

37. A 5 kg bowling ball travelling at 50% of the speed of light makes an inelastic collision with the liner Queen Mary 2 whose mass is 75 000 tons.

a) What is the speed of the ship after the collision? (The speed of the ship is small enough to use the non-relativistic formula to calculate the final momentum.)
b) What is the energy released during the collision (which corresponds to the lost kinetic energy)?
c) This released energy corresponds to how many times the energy released in the explosion of Hiroshima's atomic bomb (6.3×10^{13} J)?

38. A neutron ($m = 1.675 \times 10^{-27}$ kg) travelling at 80% of the speed of light makes an inelastic collision with a helium 3 atomic nucleus ($m = 6.646 \times 10^{-27}$ kg) at rest to form a new nucleus whose mass is 8.323×10^{-27} kg.

a) What is the speed of the new nucleus after the collision? (Assuming it is not destroyed by the collision.)
b) What is the energy released during the collision (which corresponds to the lost kinetic energy) in MeV?

39. A pion π^+ at rest, whose mass energy is 139.6 MeV, decays into two particles which move in opposite directions. One of these particles is a muon μ^+, whose mass energy is 105.7 MeV, and the other particle is a neutrino whose mass energy is very small (in this problem, we will assume that it is zero).

a) What is the kinetic energy of each particle after the decay?
b) What are the speeds of the two particles after the decay?

40. Kinetic energy, in Newton's mechanics, is given by

$$E_k = \frac{p^2}{2m}$$

Show that kinetic energy, in Einstein's mechanics, can be written in the following form.

$$E_k = \frac{p^2}{(\gamma+1)m}$$

Challenges

(Questions more difficult than the exam questions.)

41. Show that

$$\frac{1}{\sqrt{1-\frac{u'^2}{c^2}}}=\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\cdot\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\left(1-\frac{vu}{c^2}\right)$$

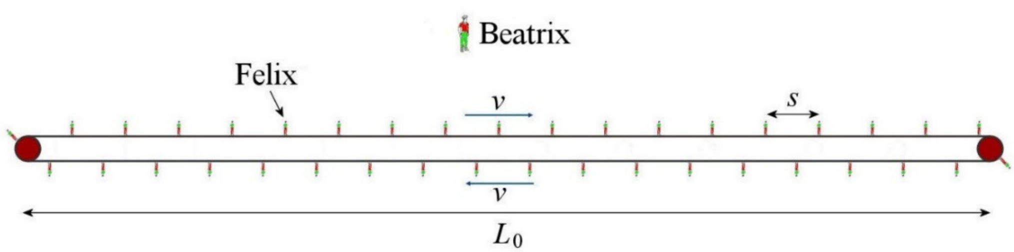
$$\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}=\frac{1}{\sqrt{1-\frac{u'^2}{c^2}}}\cdot\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\left(1+\frac{vu'}{c^2}\right)$$

42. Using the result of the previous exercise, show that, for an object moving in the x-direction, the x-component on the momentum and the relativistic energy transform according to the following formulas.

$$p'_x=\gamma\left(p_x-\frac{vE}{c^2}\right)$$

$$E'=\gamma\left(E-vp_x\right)$$

43. Beatrix is looking at a huge moving walkway moving at speed v. According to Beatrix, the distance between the ends of the walkway is L_0. On the walkway, there are N people regularly spaced at a distance s from each other (according to Beatrix). Among these people there is Felix.



a) According to Felix, how many people are on the same side of the walkway as him? (Give a result which depends only on N and v.)
b) According to Felix, how many people are on the other side of the walkway? (Give a result which depends only on N and v.) (Hint: use the result of the first challenge of this chapter.)
c) According to Felix, how many people are on the walkway?

ANSWERS

9.7 Time Dilation

1. a) 14.25 years b) 8.55 years
2. 3 min 20 s
3. Octavius is 34 years old and Augustus is 36 years old

9.8 Length Contraction

4. 3.55 cm
5. a) 0.6c b) 80 cm
6. 0.9992c
7. 0.99952c
8. 5.263
9. a) 0.8346c b) 550.8 m
10. a) 1200 m b) 5 x 10^{-6} s c) 8.333 x 10^{-6} s

9.9 Doppler Effect with Light (Take 2)

11. a) 612.6 MHz b) 15.71 MHz
12. 749.5 nm
13. 0.3133c
14. 400 GHz
15. 800 MHz
16. a) 32.03 s b) 1.601 s c) 32.03 s d) 62.45 s
17. a) 6 s b) $5c/13 = 0.3846c$

9.10 Lorentz Transformations

18. a) Explosion 2 occurred 26.67 years before explosion 1 according to Sydney.
    b) 33.33 ly
19. a) Explosion 2 occurred 30 years before explosion 1 according to Sydney.
    b) 36 ly
20. 1.2 µs
21. 0.1333 µs
22. a) 0.2c towards the left. B) 19.596 ly
23. a) 1.5 µs b) 1.375 µs c) 160 m
24. a) 3.162 ly b) $v = 0.8397c$ and $v = 0.2414c$ b) -6.245 y

9.11 Velocities Transformations

25. $-0.9987c$
26. a) $0.875c$ b) The missile does not catch Ivan
27. 90 m
28. 871.8 MHz
29. a) $0.625c$ b) Pablo c) 6 years d) 1.8735 years e) 2.4 years
30. a) 10 years b) 8 ly c) $-0.9091c$ d) 14.402 years e) 7.332 ly

9.13 Momentum and Energy

31. a) 0.0979c b) 456
32. a) 1.527×10^{-18} kgm/s b) 2070 MeV c) 3010 MeV d) 5.358×10^{-27} kg
33. a) 136.3 keV b) 321.1 keV
34. 4.26 million tons
35. 2.67×10^{-17} kgm/s
36. a) 5.118 mm b) 3.333×10^{-5} s c) 1.706×10^{-11} s
37. a) 11.55 m/s b) 6.962×10^{16} J c) 1105
38. a) $0.2592c$ b) 462 MeV
39. a) 4.1 MeV for the muon and 29.8 MeV for the neutrino.
    b) $0.2707c$ for the muon and c for the neutrino.

Challenges

43. a) $\frac{N}{2}\left(1-\frac{v^2}{c^2}\right)$ b) a) $\frac{N}{2}\left(1+\frac{v^2}{c^2}\right)$ c) N