Thick lenses

We will start by giving the formulas and then make an example. The proof of these formulas will follow.

Thick Lenses Formula

The focal length of a thick lens is given by

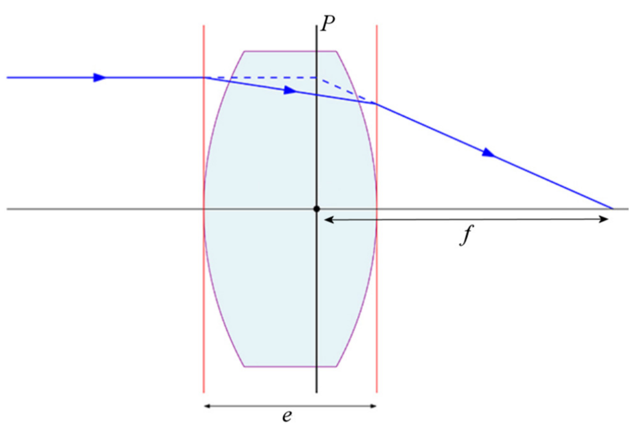
Thick Lenses Formula

$$\frac{1}{f} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{n_l - n_m}{n_m}\right) + \frac{e\left(n_l - n_m\right)^2}{n_l n_m R_1 R_2}$$

In this formula, *e* is the thickness of the lens and the other variables are identical to what they were in the thin lens formula.

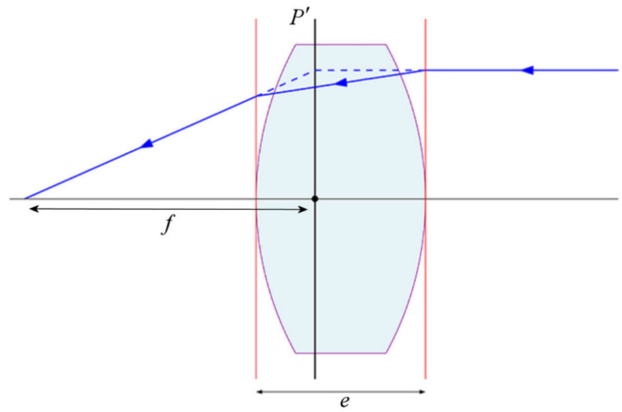
However, this focal length represents which distance exactly. Is it the distance between the focus and the center of the lens? Is it the distance between the focus and the edge of the lens? In fact, it is the distance between the focus and one of the principal planes.

Here's how the principal planes are found. When a beam parallel to the principal axis reaches a lens, it is deflected by the lens to pass through the focus. If the two rays that are outside the lens are extended, they intersect at a certain point. This crossing point corresponds to the position of one of the principal planes. The focal length is the distance between this plane and the focus.



iopscience.iop.org/book/978-0-7503-1242-4/chapter/bk978-0-7503-1242-4ch1

Another principal plane is found by using light travelling in the opposite direction, so by using a ray arriving from the left parallel to the principal axis.



The two principal planes are not in the same place, but the focal length is the same on either side of the lens.

The distance between the surface of the lens and the principal plane on the side where the light arrives is noted h_1 while the distance between the surface of the lens and the principal plane on the side where the light comes out is noted h_2.

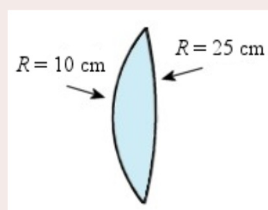
These distances are given by

Distance between the surfaces and principal planes

$$h_1 = -\frac{ef\left(n_l - n_m\right)}{n_l R_2} \qquad h_2 = -\frac{ef\left(n_l - n_m\right)}{n_l R_1}$$

The sign convention for these distances is the same as for the radius of curvature. If the value is negative, the principal plane is, relative to the surface, on the side where the light is coming. If the value is positive, the principal plane is, relative to the surface, on the side where the light is going.

Example



The lens shown in the diagram is made of a material with a refraction index of 1.5. The lens is in air.

a) What is the focal length of the lens according to the thin lens formula?

The focal length is

$$\frac{1}{f} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{n_l - n_m}{n_m}\right)$$

$$\frac{1}{f}=\left(\frac{1}{0.1m}-\frac{1}{-0.25m}\right)\left(\frac{1.5-1}{1}\right)$$

$$\frac{1}{f}=7D$$

$$f=14.286cm$$

b) What is the distance between the focus and the surfaces of the lens according to the thick lens formula if the lens thickness is 2 cm?

The focal length is

$$\frac{1}{f}=\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{n_l-n_m}{n_m}\right)+\frac{e\left(n_l-n_m\right)^2}{n_l n_m R_1 R_2}$$

$$\frac{1}{f}=\left(\frac{1}{0.1m}-\frac{1}{-0.25m}\right)\left(\frac{1.5-1}{1}\right)+\frac{0.02m\left(1.5-1\right)^2}{1.5\cdot 1\cdot 0.1m\cdot\left(-0.25m\right)}$$

$$\frac{1}{f}=7D-\tfrac{2}{15}D$$

$$f=14.563cm$$

This is the distance between the focus and the principal planes.

To the left of the lens, the distance between the lens surface and the principal plane is

$$\begin{aligned}h_1&=-\frac{ef\left(n_l-n_m\right)}{n_l R_2}\\&=-\frac{0.02m\cdot 0.1456m\left(1.5-1\right)}{1.5\cdot\left(-0.25m\right)}\\&=0.388cm\end{aligned}$$

As the value is positive, the plane is to the right (on the side where the light goes) in relation to the surface. The distance between the surface and the focus is, therefore,

$$\begin{aligned}d_{left}&=14.56cm-0.388cm\\&=14.175cm\end{aligned}$$

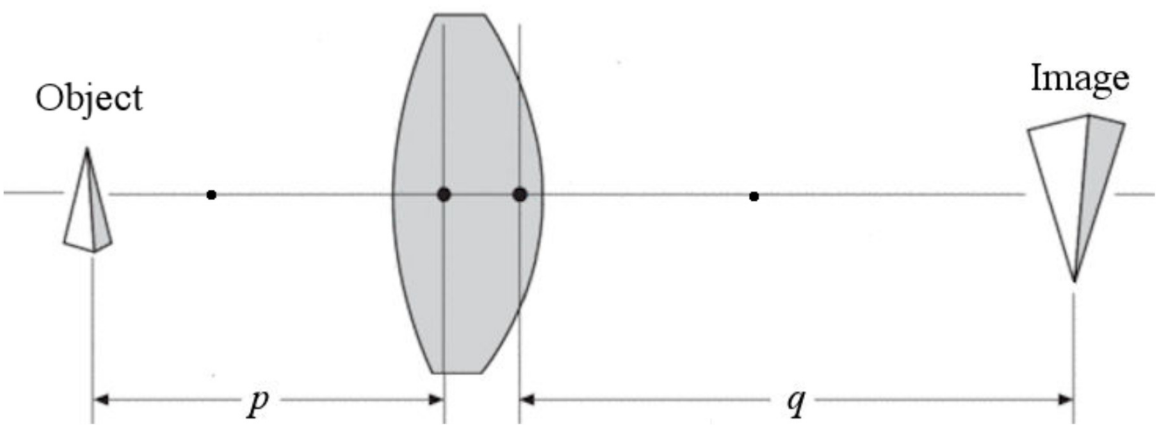
To the right of the lens, the distance between the lens surface and the principal plane is

$$h_2 = -\frac{ef\left(n_l - n_m\right)}{n_l R_1}$$
$$= -\frac{0.02m \cdot 0.1456m\left(1.5-1\right)}{1.5\cdot\left(0.10m\right)}$$
$$= -0.971cm$$

As the value is negative, the plane is to the left (on the side where the light arrives) in relation to the surface. The distance between the surface and the focus is, therefore,

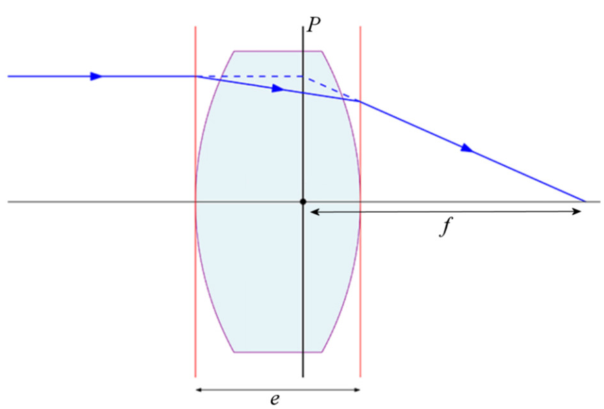
$$d_{right} = 14.563cm - 0.971cm$$
$$= 13.592cm$$

Note that in the law of lenses, the distances p and q are now measured from the principal planes.



Proof of the Formula of the Focal Length of Thick Lenses

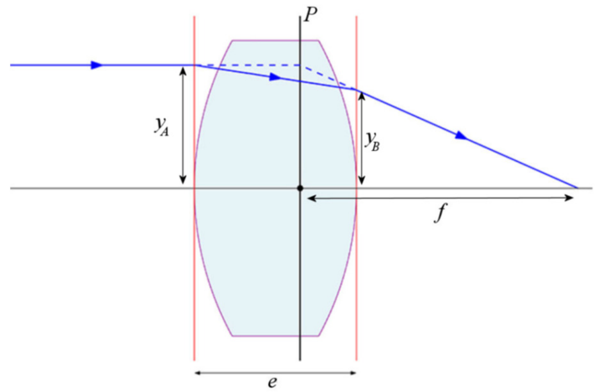
Previously, it was said that the focal length is measured from the principal plane and that this plane is found by extending the rays outside the lens to determine where they intersect. Knowing this, we will now follow the trajectory of a ray initially parallel to the principal axis through the lens to find the focal length, as shown in the diagram.



Note that we are still working with the approximation of small angles and still assuming that the rays are always near the principal axis.

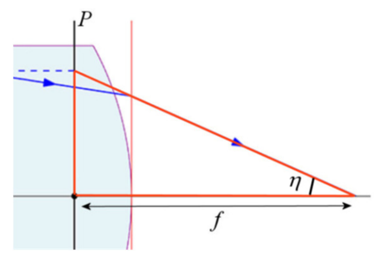
y_A is the distance between the ray and the principal axis at point A on the first surface.

y_B is the distance between the ray and the principal axis at point B on the second surface.

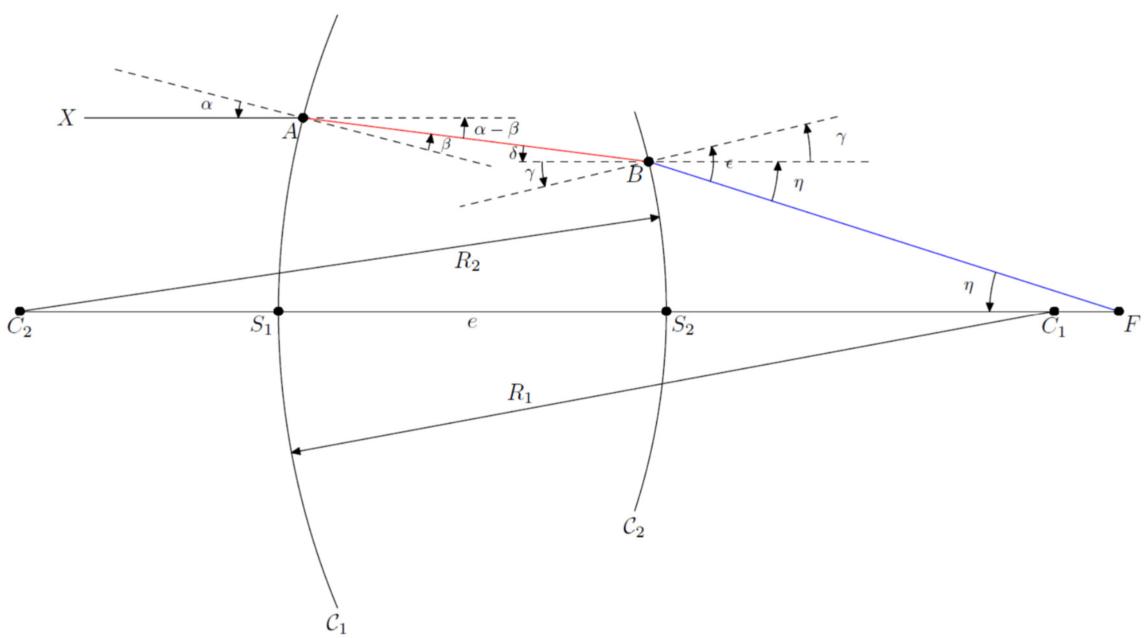


First of all, there is a triangle with the focal length. Thus, we have

$$\frac{y_A}{f} = \eta$$



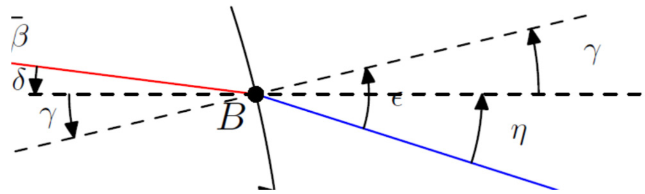
Now, we must make the link between the angle η and y_A. To achieve this, we will travel along the ray to get to the first surface. We will work with the following angles.



members.loria.fr/Roegel/loc/article-gullstrand.pdf

First, let's go to point B. We can see that $\eta = \varepsilon - \gamma$. Therefore,

$$\frac{y_A}{f} = \varepsilon - \gamma$$



The angle ε is the angle on the refracted ray. Thus, according to the law of refraction

$$n_l\left(\delta + \gamma\right) = n_m \varepsilon$$

$$\varepsilon = \frac{n_l}{n_m}\left(\delta + \gamma\right)$$

If n is defined as $n = n_l / n_m$, we can write

$$\varepsilon = n\left(\delta + \gamma\right)$$

The result is

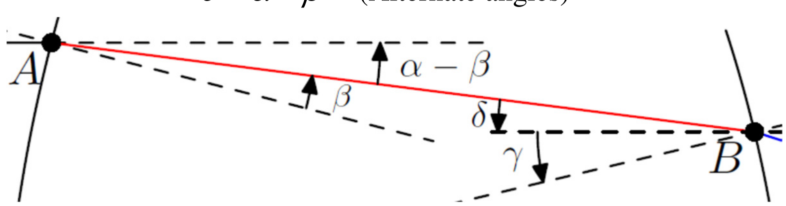
$$\frac{y_A}{f} = \varepsilon - \gamma$$

$$\frac{y_A}{f} = n\left(\delta - \gamma\right) - \gamma$$

$$\frac{y_A}{f} = n\delta + \gamma\left(n - 1\right)$$

Then, we have

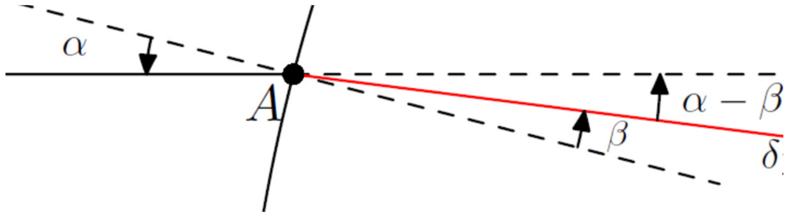
$$\delta = \alpha - \beta \quad \text{(Alternate angles)}$$



This leads us to

$$\frac{y_A}{f} = n\left(\alpha - \beta\right) + \gamma\left(n - 1\right)$$

Now let's look at what is happening at point A.



The angles α and β are the refraction angles at the first surface. According to the law of refraction, they are linked by

$$n_m\alpha = n_l\beta$$

Since $n = n_l / n_m$, this result is

$$\beta = \frac{\alpha}{n}$$

The equation then becomes

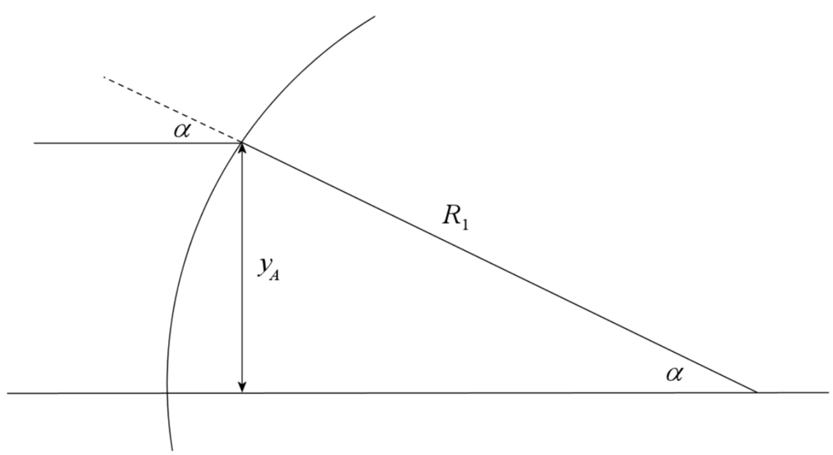
$$\frac{y_A}{f} = n\left(\alpha - \tfrac{\alpha}{n}\right) + \gamma(n-1)$$

$$\frac{y_A}{f} = n\alpha\left(1 - \tfrac{1}{n}\right) + \gamma(n-1)$$

$$\frac{y_A}{f} = \alpha(n-1) + \gamma(n-1)$$

$$\frac{y_A}{f} = (\alpha + \gamma)(n-1)$$

According to this diagram,



we have

$$\alpha \approx \frac{y_A}{R_1} \quad \text{et} \quad \gamma \approx -\frac{y_B}{R_2}$$

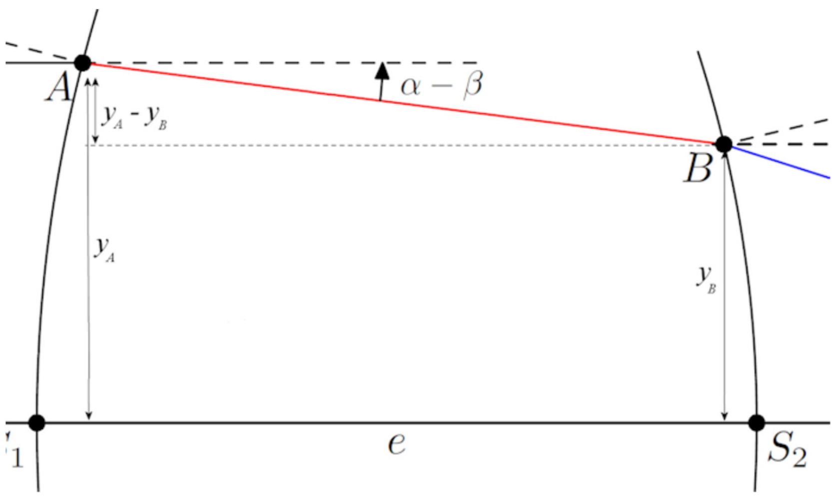
(The second equation is the same as the first, but for surface 2. However, there is a negative sign to compensate for the fact that R_2 is negative in this case.)

Thus, the equation is

$$\frac{y_A}{f} = \left(\frac{y_A}{R_1} - \frac{y_B}{R_2}\right)(n-1)$$

Now, we must make the link between y_A and y_B.

According to this diagram



we have a triangle that leads to

$$\alpha - \beta = \frac{y_A - y_B}{e}$$
$$y_A - y_B = e(\alpha - \beta)$$
$$y_B = y_A - e(\alpha - \beta)$$

However, we had

$$\beta = \frac{\alpha}{n}$$

This means that

$$y_B = y_A - e\left(\alpha - \tfrac{\alpha}{n}\right)$$
$$y_B = y_A - \alpha e\left(1 - \tfrac{1}{n}\right)$$

Using this value in

$$\frac{y_A}{f} = \left(\frac{y_A}{R_1} - \frac{y_B}{R_2}\right)(n-1)$$

The result is

$$\frac{y_A}{f} = \left(\frac{y_A}{R_1} - \frac{y_A - \alpha e\left(1 - \tfrac{1}{n}\right)}{R_2}\right)(n-1)$$

Finally, we use $\alpha = y_A / R_1$ again to arrive at

$$\frac{y_A}{f} = \left(\frac{y_A}{R_1} - \frac{y_A - \tfrac{y_A}{R_1} e\left(1 - \tfrac{1}{n}\right)}{R_2}\right)(n-1)$$
$$\frac{y_A}{f} = \left(\frac{y_A}{R_1} - \frac{y_A}{R_2} + \frac{y_A e\left(1 - \tfrac{1}{n}\right)}{R_1 R_2}\right)(n-1)$$
$$\frac{y_A}{f} = y_A\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{e\left(1 - \tfrac{1}{n}\right)}{R_1 R_2}\right)(n-1)$$
$$\frac{1}{f} = \left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{e\left(1 - \tfrac{1}{n}\right)}{R_1 R_2}\right)(n-1)$$

It is very important for the y_A to cancel out as they just have done. This shows that no matter the distance between the ray and the principal axis, the ray will reach the focus. This means that all the rays will arrive at the focus.

Let's continue to work on the equation

$$\frac{1}{f} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)(n-1) + \frac{e\left(1 - \tfrac{1}{n}\right)}{R_1 R_2}(n-1)$$
$$\frac{1}{f} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)(n-1) + \frac{e(n-1)}{n R_1 R_2}(n-1)$$
$$\frac{1}{f} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)(n-1) + \frac{e(n-1)^2}{n R_1 R_2}$$

Since $n = n_l / n_m$, the equation is

$$\frac{1}{f} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{n_l}{n_m} - 1\right) + \frac{e\left(\frac{n_l}{n_m} - 1\right)^2}{\frac{n_l}{n_m} R_1 R_2}$$

$$\frac{1}{f} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{n_l - n_m}{n_m}\right) + \frac{e\left(\frac{n_l - n_m}{n_m}\right)^2}{\frac{n_l}{n_m} R_1 R_2}$$

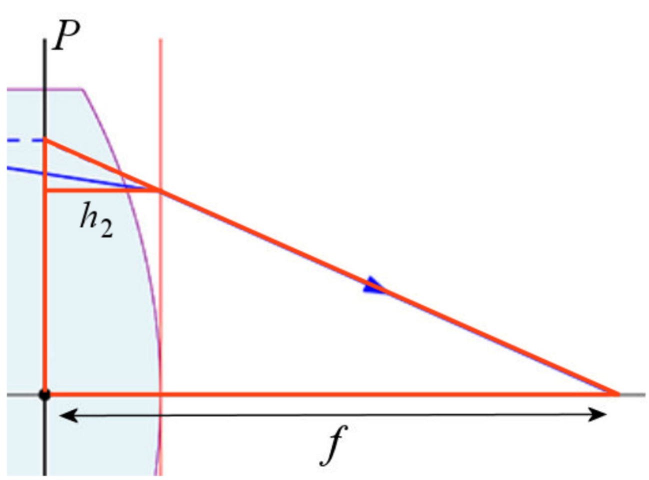
Thus, the final result is

$$\frac{1}{f} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{n_l - n_m}{n_m}\right) + \frac{e\left(n_l - n_m\right)^2}{n_l n_m R_1 R_2}$$

We recognize the part that corresponds to the thin lenses equation. The last term is the correction that takes into account the thickness of the lens.

Proof of the Formulas Giving the Position of the Principal Planes

To the right of the lens, we have two similar triangles (in red).



The height of the large triangle is y_A and the height of the small triangle is $y_A - y_B$. Therefore, the ratios of the sides of these similar triangles give

$$\frac{y_A}{f} = \frac{y_A - y_B}{h_2}$$

Previously, it had been found that

$$y_A - y_B = \alpha e\left(1 - \tfrac{1}{n}\right)$$

Thus

$$\frac{y_A}{f} = \frac{\alpha e\left(1 - \tfrac{1}{n}\right)}{h_2}$$

It was also found that $\alpha = y_A / R_1$. Thus, the equation becomes

$$\frac{y_A}{f} = \frac{y_A e\left(1 - \tfrac{1}{n}\right)}{Rh_2}$$

Therefore,

$$\frac{1}{f} = \frac{e\left(1 - \tfrac{1}{n}\right)}{R_1 h_2}$$

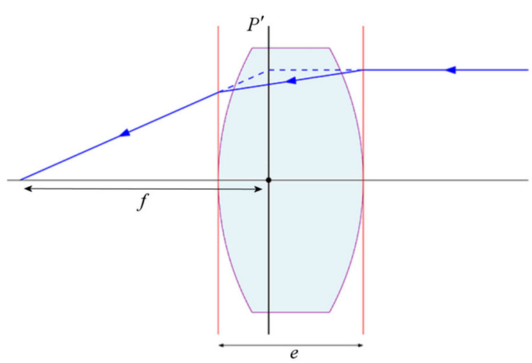
$$h_2 = \frac{ef\left(1 - \tfrac{1}{n}\right)}{R_1}$$

$$h_2 = \frac{ef\left(1 - \tfrac{n_m}{n_l}\right)}{R_1}$$

$$h_2 = \frac{ef\left(n_l - n_m\right)}{n_l R_1}$$

It only remains to add a negative sign to respect the sign convention

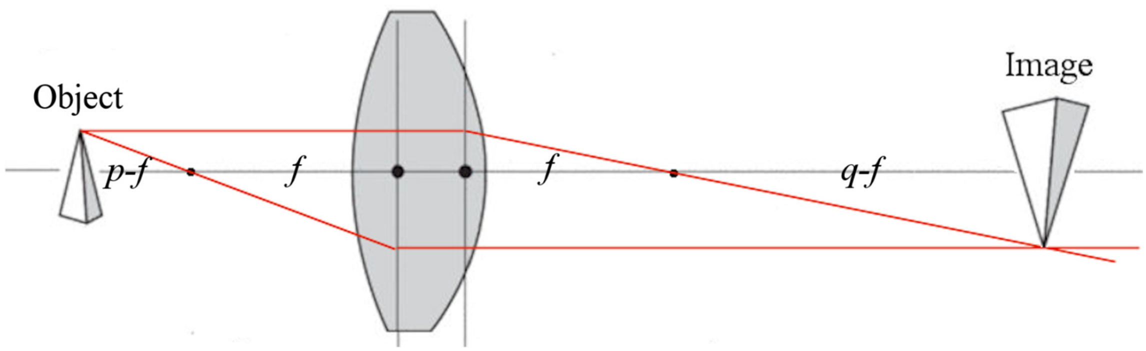
The proof for h_1 is done the same way on the other side of the lens with this ray.



Proof that the Positions of the Object and the Image Must Be Taken From the Principal Planes

We will assume that distances are measured from the principal planes. We'll then see that this assumption leads to the correct lens equation.

These two rays are used. (They are deflected at the principal planes. The foci are both at the same distance from the principal planes.)



On the left, there are two similar triangles (they have opposite angles the focus). The ratios on the sides give

$$\frac{y_o}{y_i} = \frac{p-f}{f}$$

On the right, there are two similar triangles (they have opposite angles the focus). The ratios on the sides give

$$\frac{y_o}{y_i} = \frac{f}{q-f}$$

Thus

$$\frac{p-f}{f} = \frac{f}{q-f}$$

This equation leads to

$$\frac{p-f}{f}=\frac{f}{q-f}$$

$$\frac{p}{f}-\frac{f}{f}=\frac{f}{q-f}$$

$$\frac{p}{f}-1=\frac{f}{q-f}$$

$$\frac{p}{f}=\frac{f}{q-f}+1$$

$$\frac{p}{f}=\frac{f}{q-f}+\frac{q-f}{q-f}$$

$$\frac{p}{f}=\frac{f+q-f}{q-f}$$

$$\frac{p}{f}=\frac{q}{q-f}$$

$$\frac{f}{p}=\frac{q-f}{q}$$

$$\frac{f}{p}=\frac{q}{q}-\frac{f}{q}$$

$$\frac{f}{p}=1-\frac{f}{q}$$

$$\frac{1}{p}=\frac{1}{f}-\frac{1}{q}$$

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$$

Voilà.