Chapter 8 Solutions

1. a) The duration of the trip is

$$\begin{aligned}\Delta t &= \frac{d}{v}\\ &= \frac{11.4ly}{0.8c}\\ &= \frac{11.4y\cdot c}{0.8c}\\ &= \frac{11.4y}{0.8}\\ &= 14.25y\end{aligned}$$

b) The duration of the trip is found with the time dilation formula. It's Adolf who measures the proper time (Δt_0) since he is the one present at both events.

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$14.25y = \frac{\Delta t_0}{\sqrt{1-\frac{(0.8c)^2}{c^2}}}$$

$$14.25y = \frac{\Delta t_0}{\sqrt{1-(0.8)^2}}$$

$$\Delta t_0 = 8.55y$$

2. The motion will be split into two steps performed at a constant velocity: going away and returning.

While going away, the time on Earth's clock is

$$\Delta t = \frac{d}{v} = \frac{9\times 10^{10}m}{0.6\cdot 3\times 10^{8}\,\frac{m}{s}} = 500s$$

The moving clock is the clock measuring the proper time. Therefore

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$500s = \frac{\Delta t_0}{\sqrt{1-\frac{(0.6c)^2}{c^2}}}$$

$$500s = \frac{\Delta t_0}{\sqrt{1-(0.6)^2}}$$

$$\Delta t_0 = 400s$$

During the return trip, the clock in motion still travels 90 million km at 0.6*c*. The time, according to observers on Earth, is

$$\Delta t = \frac{d}{v} = \frac{9\times 10^{10} m}{0.6\cdot 3\times 10^8 \frac{m}{s}} = 500s$$

The moving clock is still the clock measuring the proper time. Therefore

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$500s = \frac{\Delta t_0}{\sqrt{1-\frac{(0.6c)^2}{c^2}}}$$

$$500s = \frac{\Delta t_0}{\sqrt{1-(0.6)^2}}$$

$$\Delta t_0 = 400s$$

So, the clock on Earth has advanced by 1000s and the clock in motion has advanced by 800 s. The clock in motion is therefore late by 200 s = 3 min. 20 sec.

3. Let's find the duration of Augustus's trip. For an observer on Earth, Augustus travels 12 ly at a speed of 0.6*c*. The duration of the trip according to the observer on Earth is

$$\begin{aligned}\Delta t &= \frac{12ly}{0.6c} \\ &= \frac{12y\cdot c}{0.6c} \\ &= 20y\end{aligned}$$

For Augustus, the duration of the trip is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$20y = \frac{\Delta t_0}{\sqrt{1-\frac{(0,6c)^2}{c^2}}}$$

$$20y = \frac{\Delta t_0}{\sqrt{1-(0,6)^2}}$$

$$\Delta t_0 = 16y$$

Augustus is, therefore, 36-year-old when he arrives on the planet.

Now let's look at Octavius's journey. For an observer on Earth, Octavius travels 12 ly at a speed of 0.8*c*. The duration of the trip according to the observer on Earth is

$$\Delta t = \frac{12ly}{0.8c}$$

$$= \frac{12y \cdot c}{0.8c}$$

$$= 15y$$

For Octavius, the duration is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$15y = \frac{\Delta t_0}{\sqrt{1-\frac{(0.8c)^2}{c^2}}}$$

$$15y = \frac{\Delta t_0}{\sqrt{1-(0.8)^2}}$$

$$\Delta t_0 = 9y$$

However, Octavius has to wait for the arrival of his brother. For the inhabitants of the planet (who are in the same reference frame as the Earth), the time interval between the arrival of the two brothers is 5 years (20 years for Augustus 15 for Octavius). Octavius must wait 5 years before the arrival of his brother. So he has aged 14 years (9 years for the trip and 5 years of waiting) when Augustus arrives.

Augustus's age is, therefore: 20 years + 16 years = 36 years

Octavius's age is, therefore: 20 years + 14 years = 34 years

The day of Augustus's arrival corresponds to the birthday of both twins, Octavius celebrates his 34th birthday and Augustus celebrates his 36th birthday.

4. The width of the ball is

$$\begin{aligned} L &= L_0\sqrt{1-\tfrac{v^2}{c^2}} \\ &= 7.2cm\cdot\sqrt{1-\tfrac{(0.87c)^2}{c^2}} \\ &= 7.2cm\cdot\sqrt{1-(0.87)^2} \\ &= 3.55cm \end{aligned}$$

5. a) Tom sees a 100 cm long ruler whose length is only 80 cm. Therefore, the speed is

$$\begin{aligned} L &= L_0\sqrt{1-\tfrac{v^2}{c^2}} \\ 80cm &= 100cm\cdot\sqrt{1-\tfrac{v^2}{c^2}} \\ 0.8 &= \sqrt{1-\tfrac{v^2}{c^2}} \\ 0.64 &= 1-\tfrac{v^2}{c^2} \\ \tfrac{v^2}{c^2} &= 0.36 \\ \tfrac{v}{c} &= 0.6 \\ v &= 0.6c \end{aligned}$$

b) For the alien, the 100 cm long ruler held by Tom moves at 0.6*c*. The length of the rule is then

$$\begin{aligned} L &= L_0\sqrt{1-\tfrac{v^2}{c^2}} \\ &= 100cm\cdot\sqrt{1-\tfrac{(0.6c)^2}{c^2}} \\ &= 80cm \end{aligned}$$

6. The speed is found with

$$L = L_0\sqrt{1-\tfrac{v^2}{c^2}}$$
$$20ly = 500ly \cdot \sqrt{1-\tfrac{v^2}{c^2}}$$
$$0.04 = \sqrt{1-\tfrac{v^2}{c^2}}$$
$$0.0016 = 1-\tfrac{v^2}{c^2}$$
$$\tfrac{v^2}{c^2} = 0.9984$$
$$\tfrac{v}{c} = 0.9992$$
$$v = 0.9992c$$

7. The duration of the trip, according to an observer on Earth, is

$$\Delta t = \frac{L_0}{v}$$

The time according to Raoul (who measures the proper time) can then be calculated with

$$\Delta t_0 = \Delta t\sqrt{1-\frac{v^2}{c^2}}$$
$$\Delta t_0 = \frac{L_0}{v}\sqrt{1-\frac{v^2}{c^2}}$$

If the trip is to last 20 years, the following equation must be solved

$$20y = \frac{643y \cdot c}{v}\sqrt{1-\frac{v^2}{c^2}}$$

This leads to

$$\frac{20}{643} = \frac{c}{v}\sqrt{1-\frac{v^2}{c^2}}$$
$$\left(\frac{20}{643}\right)^2 = \frac{c^2}{v^2}\left(1-\frac{v^2}{c^2}\right)$$
$$\left(\frac{20}{643}\right)^2 = \frac{c^2}{v^2}-1$$

$$\left(\frac{20}{643}\right)^2 + 1 = \frac{c^2}{v^2}$$

$$v^2 = \frac{1}{\left(\frac{20}{643}\right)^2 + 1} c^2$$

$$v = \frac{1}{\sqrt{\left(\frac{20}{643}\right)^2 + 1}} c$$

$$v = 0{,}99952c$$

The speed must be 99,952% of the speed of light.

8. The length of vessels according to Ahmed will be denoted *x*. Since Yitzhak's spaceship is contracted, its length is

$$L_0 = \frac{x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This is the length of Yitzhak's spaceship according to Yitzhak.

For Yitzhak, Ahmed's spaceship, whose length is *x* at rest, is contracted. Its length is therefore

$$L = x\sqrt{1 - \frac{v^2}{c^2}}$$

The ratio of lengths according to Yitzhak is, therefore,

$$\begin{aligned}\frac{L_{Yitzhak}}{L_{Ahmed}} &= \frac{x / \sqrt{1 - \frac{v^2}{c^2}}}{x\sqrt{1 - \frac{v^2}{c^2}}} \\ &= \frac{1}{1 - \frac{v^2}{c^2}} \\ &= \frac{1}{1 - 0.9^2} \\ &= 5.263\end{aligned}$$

9. a) The duration of the trip according to an observer on Earth is

$$\Delta t = \frac{L_0}{v} = \frac{1000m}{v}$$

The duration in the muon's reference frame (which is the proper time) can then be calculated with

$$\Delta t_0 = \Delta t \sqrt{1-\frac{v^2}{c^2}}$$
$$\Delta t_0 = \frac{L_0}{v} \sqrt{1-\frac{v^2}{c^2}}$$

If the trip is to last 2.2 μs in the muon's reference frame, the following equation must be solved

$$2.2\mu s = \frac{1000m}{v} \cdot \sqrt{1-\frac{v^2}{c^2}}$$

This leads to

$$\frac{2.2\mu s}{1000m} = \frac{1}{v}\sqrt{1-\frac{v^2}{c^2}}$$
$$\left(\frac{2.2\mu s}{1000m}\right)^2 = \frac{1}{v^2}\left(1-\frac{v^2}{c^2}\right)$$
$$\left(\frac{2.2\mu s}{1000m}\right)^2 = \frac{1}{v^2}-\frac{1}{c^2}$$
$$4.84\times 10^{-18} \tfrac{s^2}{m^2} = \frac{1}{v^2}-\frac{1}{c^2}$$
$$4.84\times 10^{-18} \tfrac{s^2}{m^2} + \frac{1}{c^2} = \frac{1}{v^2}$$
$$1.59511\times 10^{-17} \tfrac{s^2}{m^2} = \frac{1}{v^2}$$
$$v = 2.504\times 10^{8} \tfrac{m}{s} = 0.8346c$$

b) In the muon's reference frame, the tunnel is moving at 0.8346c. Its length is therefore

$$L = L_0\sqrt{1-\tfrac{v^2}{c^2}}$$
$$= 1000m \cdot \sqrt{1-\tfrac{(0.8346c)^2}{c^2}}$$
$$= 1000m \cdot \sqrt{1-(0.8346)^2}$$
$$= 550.8m$$

10. a) The length of the train is

$$\begin{aligned} L &= L_0\sqrt{1-\tfrac{v^2}{c^2}} \\ &= 2000m\cdot\sqrt{1-\tfrac{(0.8c)^2}{c^2}} \\ &= 2000m\cdot\sqrt{1-(0.8)^2} \\ &= 1200m \end{aligned}$$

b) According to Agathe, the train must move forward 1200 m. Therefore, the time is

$$\Delta t_0 = \frac{d}{v} = \frac{1200m}{0.8c} = 5\times10^{-6}s$$

This is the proper time because Agathe is present at both events. In other words, the two events occur at the same place (besides the pole) in her reference frame. In the reference frame of the train, an event occurs at the front of the train (start of the stopwatches) while the other event (the stopwatches stop) is at the rear of the train.

c) The time according to Justin is

$$\begin{aligned} \Delta t &= \frac{\Delta t_0}{\sqrt{1-\tfrac{v^2}{c^2}}} \\ &= \frac{5\times10^{-6}s}{\sqrt{1-\tfrac{(0.8c)^2}{c^2}}} \\ &= \frac{5\times10^{-6}s}{\sqrt{1-(0.8)^2}} \\ &= 8.333\times10^{-6}s \end{aligned}$$

Note:

The time can also be found by saying that, for Justin, the post must pass from the front of the train to the rear of the train (a distance of 2000 m since the train is not contracted according to Justin) at the speed of 0.8*c*. The time is therefore

$$\Delta t = \frac{d}{v} = \frac{2000m}{0.8c} = 8.333\times10^{-6}s$$

11. a) The received frequency is

$$\begin{aligned} f' &= f_0\sqrt{\frac{c+v}{c-v}} \\ &= 98.1MHz \cdot \sqrt{\frac{c+0.95c}{c-0.95c}} \\ &= 98.1MHz \cdot \sqrt{\frac{1+0.95}{1-0.95}} \\ &= 612.6MHz \end{aligned}$$

b) The received frequency is

$$\begin{aligned} f' &= f_0\sqrt{\frac{c+v}{c-v}} \\ &= 98.1MHz \cdot \sqrt{\frac{c+-0.95c}{c--0.95c}} \\ &= 98.1MHz \cdot \sqrt{\frac{1-0.95}{1+0.95}} \\ &= 15.7MHz \end{aligned}$$

12. We do not have a formula for the change of wavelength but we have one for the change of frequency. Thus, we will find the frequency from the wavelength, use the Doppler shift formula to calculate the new frequency, and finally calculate the final wavelength from the shifted frequency.

A 550 nm wavelength corresponds to a frequency of

$$f = \frac{c}{\lambda} = \frac{3\times 10^8 \frac{m}{s}}{550\times 10^{-9} m} = 5.4545\times 10^{14} Hz$$

The received frequency is, therefore,

$$\begin{aligned} f' &= f_0\sqrt{\frac{c+v}{c-v}} \\ &= 5.4545\times10^{14}\,Hz\cdot\sqrt{\frac{c+-0.3c}{c--0.3c}} \\ &= 5.4545\times10^{14}\,Hz\cdot\sqrt{\frac{1-0.3}{1+0.3}} \\ &= 4.0025\times10^{14}\,Hz \end{aligned}$$

This corresponds to a wavelength of

$$\lambda = \frac{c}{f} = \frac{3\times10^{8}\,\frac{m}{s}}{4.0025\times10^{14}\,Hz} = 749.5\times10^{-9}\,m = 749.5nm$$

13. We do not have a formula for the change of wavelength but we have one for the change of frequency. Thus, we will find the frequencies from the wavelengths and then use the Doppler shift formula to calculate the speed.

A 650 nm wavelength corresponds to a frequency of

$$f = \frac{c}{\lambda} = \frac{3\times10^{8}\,\frac{m}{s}}{650\times10^{-9}\,m} = 4.615\times10^{14}\,Hz$$

A 470 nm wavelength corresponds to a frequency of

$$f = \frac{c}{\lambda} = \frac{3\times10^{8}\,\frac{m}{s}}{470\times10^{-9}\,m} = 6.383\times10^{14}\,Hz$$

Thus, the speed is found with

$$\begin{aligned} f' &= f_0\sqrt{\frac{c+v}{c-v}} \\ 6.383\times10^{14}\,Hz &= 4.615\times10^{14}\,Hz\cdot\sqrt{\frac{c+v}{c-v}} \\ 1.383 &= \sqrt{\frac{c+v}{c-v}} \\ 1.9126 &= \frac{c+v}{c-v} \\ 1.9126c - 1.9126\cdot v &= c+v \end{aligned}$$

$$0.9126c = 2.9126 \cdot v$$
$$v = \frac{0.9126c}{2.9126}$$
$$v = 0.3133c$$

14. The frequency received by the spaceship is

$$\begin{aligned} f' &= f_0 \sqrt{\frac{c+v}{c-v}} \\ &= 100GHz \cdot \sqrt{\frac{c+0.6c}{c-0.6c}} \\ &= 100GHz \cdot \sqrt{\frac{1+0.6}{1-0.6}} \\ &= 200GHz \end{aligned}$$

Through reflection, the spaceship therefore emits a 200 GHz wave. Now, there is a 200 GHz source heading towards the Earth at 0.6*c*. Therefore, the frequency received on Earth is

$$\begin{aligned} f' &= f_0 \sqrt{\frac{c+v}{c-v}} \\ &= 200GHz \cdot \sqrt{\frac{c+0.6c}{c-0.6c}} \\ &= 200GHz \cdot \sqrt{\frac{1+0.6}{1-0.6}} \\ &= 400GHz \end{aligned}$$

15. First, let's find the speed of the spaceship. The spaceship and Raphael will first be considered. The positive axis is towards the left since the wave goes from the spaceship towards Raphael. As the spaceship travels towards the right, its speed is negative. Therefore

$$f' = f_0 \sqrt{\frac{c+(-v)}{c-(-v)}}$$
$$200MHz = 400MHz \cdot \sqrt{\frac{c-v}{c+v}}$$

$$0.5 = \sqrt{\frac{c-v}{c+v}}$$

$$0.25 = \frac{c-v}{c+v}$$

$$0.25c + 0.25 \cdot v = c - v$$

$$1.25 \cdot v = 0.75c$$

$$v = \frac{0.75c}{1.25}$$

$$v = 0.6c$$

Now, we will consider what is happening with William. In this case, the wave travels towards the right (from the spaceship to William). As the spaceship also travels towards the right, the speed of the ship is positive. The frequency received by William is, therefore,

$$\begin{aligned} f' &= f_0 \sqrt{\frac{c+v}{c-v}} \\ &= 400MHz \cdot \sqrt{\frac{c+0.6c}{c-0.6c}} \\ &= 400MHz \cdot \sqrt{\frac{1+0.6}{1-0.6}} \\ &= 800MHz \end{aligned}$$

16. a)

10 seconds is the proper time between the flashes (because it is the time measured by Ursula, who is present at every flash). Since the spaceship is moving at $0.95c$ according to Wilfrid, the time between the flashes is

$$\begin{aligned} \Delta t &= \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}} \\ &= \frac{10s}{\sqrt{1-(0.95)^2}} \\ &= 32.03s \end{aligned}$$

b) The wave received by Wilfrid moves towards the right, and the spaceship also travels towards the right. This means that the speed of the ship is positive in the Doppler effect formula. Therefore, the time between the flashes

$$
\begin{aligned}
T' &= T_0\sqrt{\frac{c-v}{c+v}} \\
&= 10s\cdot\sqrt{\frac{c-0.95c}{c+0.95c}} \\
&= 10s\cdot\sqrt{\frac{0.05}{1.95}} \\
&= 1.601s
\end{aligned}
$$

c) 10 seconds is the proper time between the flashes (because it is the time measured by Ursula, who is present at every flash). Since the spaceship is moving at 0.95c according to Flavien, the time between the flashes is

$$
\begin{aligned}
\Delta t &= \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}} \\
&= \frac{10s}{\sqrt{1-(0.95)^2}} \\
&= 32.03s
\end{aligned}
$$

d) The wave received by Flavien travels towards the left and the spaceship travels towards the right. This means that the speed of the ship is negative in the Doppler effect formula. Therefore, the time between the flashes

$$
\begin{aligned}
T' &= T_0\sqrt{\frac{c-v}{c+v}} \\
&= 10s\cdot\sqrt{\frac{c--0.95c}{c+-0.95c}} \\
&= 10s\cdot\sqrt{\frac{1.95}{0.05}} \\
&= 62.45s
\end{aligned}
$$

17. Flavien sees the ship moving away. This means that the time between flashes according to Flavien is

$$T' = T_0\sqrt{\frac{c-(-v)}{c+(-v)}}$$
$$T' = T_0\sqrt{\frac{c+v}{c-v}}$$

Since this time is 9 seconds according to Flavien, this first equation is obtained.

$$9s = T_0\sqrt{\frac{c+v}{c-v}}$$

Wilfrid sees the ship approaching. This means that the time between flashes according to Wilfrid is

$$T' = T_0\sqrt{\frac{c-v}{c+v}}$$

Since this time is 4 seconds according to Wilfrid, this second equation is obtained

$$4s = T_0\sqrt{\frac{c-v}{c+v}}$$

Thus, we have 2 equations with 2 unknowns. The period of the signal according to Ursula can be found by multiplying the two equations

$$4s\cdot 9s = T_0\sqrt{\frac{c-v}{c+v}}\cdot T_0\sqrt{\frac{c+v}{c-v}}$$
$$36s^2 = T_0^2$$
$$T_0 = 6s$$

The speed can be found by dividing the two equations.

$$\frac{9s}{4s} = \frac{T_0\sqrt{\frac{c+v}{c-v}}}{T_0\sqrt{\frac{c-v}{c+v}}}$$
$$\frac{9}{4} = \frac{c+v}{c-v}$$
$$9(c-v) = 4(c+v)$$
$$9c-9v = 4c+4v$$

$$5c = 13v$$
$$v = \frac{5c}{13}$$

18. According to Gertrude, we have

$$\Delta t = 0$$
$$\Delta x = 20ly$$

a) Therefore, the time between explosions according to Sydney is

$$\begin{aligned}
\Delta t' &= \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \\
&= \frac{1}{\sqrt{1-\frac{(0.8c)^2}{c^2}}}\cdot\left(0-\frac{0.8c\cdot 20ly}{c^2}\right) \\
&= \frac{1}{\sqrt{1-(0.8)^2}}\cdot\left(0-\frac{0.8\not{c}\cdot 20y\not{c}}{\not{c}^2}\right) \\
&= \frac{1}{\sqrt{1-(0.8)^2}}\cdot(0-0.8\cdot 20y) \\
&= -26.67y
\end{aligned}$$

A negative answer means that $t_2 - t_1 < 0$ which means that t_2 is smaller than t_1. Explosion 2 therefore occurred 26.67 years before the explosion 1 occurred according to Sydney.

b) The distance between the places where the explosions occurred according to Sidney is

$$\begin{aligned}
\Delta x' &= \gamma(\Delta x - v\Delta t) \\
&= \frac{1}{\sqrt{1-\frac{(0.8c)^2}{c^2}}}\cdot(20ly - 0.8c\cdot 0y) \\
&= \frac{1}{\sqrt{1-(0.8)^2}}\cdot(20ly) \\
&= 33.33ly
\end{aligned}$$

19. According to Gertrude, we have

$$\Delta t = t_2 - t_1 = -2y$$
$$\Delta x = 20ly$$

a) Therefore, the time between explosions according to Sydney is

$$\begin{aligned}\Delta t' &= \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)\\ &= \frac{1}{\sqrt{1-\frac{(0.8c)^2}{c^2}}}\cdot\left((-2y) - \frac{0.8c\cdot 20ly}{c^2}\right)\\ &= \frac{1}{\sqrt{1-(0.8)^2}}\cdot\left(-2y - \frac{0,8\cancel{c}\cdot 20y\cancel{c}}{\cancel{c^2}}\right)\\ &= \frac{1}{\sqrt{1-(0.8)^2}}\cdot(-2y - 0.8\cdot 20y)\\ &= -30y\end{aligned}$$

A negative answer means that $t_2 - t_1 < 0$ which means that t_2 is smaller than t_1. Explosion therefore occurred 30 years before the explosion 1 occurred according to Sydney.

b) The distance between the places where the explosions occurred according to Sidney is

$$\begin{aligned}\Delta x' &= \gamma(\Delta x - v\Delta t)\\ &= \frac{1}{\sqrt{1-\frac{(0.8c)^2}{c^2}}}\cdot(20ly - 0.8c\cdot(-2y))\\ &= \frac{1}{\sqrt{1-(0.8)^2}}\cdot(21.6ly)\\ &= 36ly\end{aligned}$$

20. According to Rodolphe, the ray travels 120 m from the rear to the front of the train. So, we have

$$\Delta x' = 120m$$
$$\Delta t' = \frac{120m}{3\times10^8 \frac{m}{s}} = 4\times10^{-7} s$$

Therefore, the time according to Jean-Marie is

$$\begin{aligned}\Delta t &= \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)\\ &= \frac{1}{\sqrt{1-\frac{(0.8c)^2}{c^2}}}\cdot\left(4\times10^{-7} s + \frac{0.8c\cdot 120m}{c^2}\right)\\ &= \frac{1}{\sqrt{1-(0.8)^2}}\cdot\left(4\times10^{-7} s + \frac{0.8\cdot 120m}{c}\right)\\ &= \frac{5}{3}\cdot\left(7.2\times10^{-7} s\right)\\ &= 1.2\times10^{-6} s\end{aligned}$$

Note:

The answer can also be found by taking the point of view of the Jean-Marie. For Jean-Marie, the length of the train is 72 m. The light, which leaves the rear of the train, has to catch up with the front of the train which moves to 0.8*c*. The time is, therefore,

$$\begin{aligned}\Delta t &= \frac{L}{v_1 - v_2}\\ &= \frac{72m}{c-0.8c}\\ &= 1.2\times10^{-6} s\end{aligned}$$

(The formula

$$\Delta t = \frac{L}{v_1 - v_2}$$

was used. This formula, seen in chapter 1 in mechanics, is used to find in how long it will take before two objects that are going at a constant speed meet.)

21. According to Rodolphe, the ray travels 120 m from the front to the rear of the train. So, we have

$$\Delta x' = -120m$$
$$\Delta t' = \frac{120m}{3\times10^8 \frac{m}{s}} = 4\times10^{-7} s$$

Therefore, the time according to Jean-Marie is

$$\begin{aligned}\Delta t &= \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)\\ &= \frac{1}{\sqrt{1-\frac{(0.8c)^2}{c^2}}}\cdot\left(4\times10^{-7} s + \frac{0.8c\cdot(-120m)}{c^2}\right)\\ &= \frac{1}{\sqrt{1-(0.8)^2}}\cdot\left(4\times10^{-7} s - \frac{0.8\cdot 120m}{c}\right)\\ &= \frac{5}{3}\cdot\left(8\times10^{-8} s\right)\\ &= 1.333\times10^{-7} s\end{aligned}$$

Note:

The answer can also be found by taking the point of view of the Jean-Marie. For Jean-Marie, the length of the train is 72 m. The light, which leaves the front of the train, is going head to head with the back of the train which moves to 0.8*c*. The time is, therefore,

$$\begin{aligned}\Delta t &= \frac{L}{v_1 - v_2}\\ &= \frac{72m}{c - -0.8c}\\ &= 1.333\times10^{-7} s\end{aligned}$$

(The formula

$$\Delta t = \frac{L}{v_1 - v_2}$$

was used. This formula, seen in chapter 1 in mechanics, is used to find in how long it will take before two objects that are going at a constant speed meet.)

22. a) According to Gertrude, we have

$$\Delta x = 20ly$$
$$\Delta t = -4y$$

It was assumed that Sydney is going in the direction shown in the figure. If a negative velocity is obtained, then this assumption was not correct.

In order to have simultaneous explosions according to Sydney, we must have

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)$$
$$0y = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\left((-4y) - \frac{v\cdot 20ly}{c^2}\right)$$
$$0y = \left(-4y - \frac{v\cdot 20y\cdot c}{c^2}\right)$$
$$0y = \left(-4y - \frac{v\cdot 20y}{c}\right)$$
$$-4y = \frac{v\cdot 20y}{c}$$
$$\frac{v}{c} = -0.2$$
$$v = -0.2c$$

As the answer is negative, the assumption was not correct. Sydney therefore travels towards the left at $0.2c$.

b) Here are 2 ways to solve this problem.

1) We can find the distance with Lorentz transformations. Since we obtained a negative velocity v in a), Gertrude sees Sydney going towards the negatives x's. Therefore, Gertrude should be the one using primes. So, we have

$$\Delta x' = 20al$$
$$\Delta t' = -4a$$

The distance according to Sydney is therefore

$$\begin{aligned}\Delta x &= \gamma\left(\Delta x' + v\Delta t'\right)\\ &= \frac{1}{\sqrt{1-\frac{(0.2c)^2}{c^2}}}\cdot\left(20y\cdot c + 0,2c\cdot(-4y)\right)\\ &= \frac{1}{\sqrt{1-(0.2)^2}}\cdot(19.2yc)\\ &= 19.596ly\end{aligned}$$

2) The interval can also be used. With Gertrude's data, we have

$$\begin{aligned}I' &= \left(c\Delta t'\right)^2 - \left(\Delta x'\right)^2\\ &= \left(c\cdot -4y\right)^2 - \left(20ly\right)^2\\ &= 16y^2\cdot c^2 - 400ly^2\\ &= -384ly^2\end{aligned}$$

We can now find the proper distance with

$$\begin{aligned}\Delta\sigma &= \sqrt{-I}\\ &= \sqrt{--384ly^2}\\ &= \sqrt{384ly^2}\\ &= 19.596ly\end{aligned}$$

We take the positive root, because $\Delta\sigma$ must have the same sign as the Δx used to calculate the interval (since the order in x is the same for all observers if $I < 0$).

23. a)

According to Tatiana, the two events are 200 m one from the other. The time between events is the time it takes for the rear of the train to arrive at the second post. To get there, the back of the train must travel 270 m to 0.6c. The time is, therefore,

$$\Delta t = \frac{270m}{0.6c} = 1.5\times 10^{-6}s$$

b) According to Tatiana, we have

$$\Delta x = 200m$$

$$\Delta t = 1.5\times 10^{-6}s$$

Therefore, the time according to Serge is

$$\begin{aligned}\Delta t' &= \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)\\ &= \frac{1}{\sqrt{1-\frac{(0.6c)^2}{c^2}}}\cdot\left(1.5\times10^{-6}s - \frac{0.6c\cdot 200m}{c^2}\right)\\ &= \frac{1}{\sqrt{1-(0.6)^2}}\cdot\left(1.5\times10^{-6}s - \frac{0.6\cdot 200m}{c}\right)\\ &= \frac{5}{4}\cdot\left(1.1\times10^{-6}s\right)\\ &= 1.375\times10^{-6}s\end{aligned}$$

c) The distance between the posts according to Serge is

$$\begin{aligned}L &= L_0\sqrt{1-\tfrac{v^2}{c^2}}\\ &= 200m\cdot\sqrt{1-\tfrac{(0.6c)^2}{c^2}}\\ &= 200m\cdot\sqrt{1-(0.6)^2}\\ &= 160m\end{aligned}$$

<u>*Notes:*</u>

The formula

$$\Delta x' = \gamma\left(\Delta x + v\Delta t\right)$$

cannot be used here because this formula will give us the distance between the two events according to Serge. The first event is the first post besides the front of the train, and the second event is the second pole besides the rear of the train. So there is an event at the front of the train and another at the rear of the train. The distance between the two events is simply the length of the train according to Serge. Let's check it. The length of the train according to Serge is

$$L = L_0\sqrt{1-\tfrac{v^2}{c^2}}$$
$$70m = L_0\sqrt{1-\tfrac{(0.6c)^2}{c^2}}$$
$$70m = L_0\sqrt{1-(0.6)^2}$$
$$L_0 = 87.5m$$

while the Lorentz transformation gives

$$\begin{aligned}\Delta x' &= \gamma(\Delta x - v\Delta t)\\ &= \frac{1}{\sqrt{1-\frac{(0.6c)^2}{c^2}}}\cdot\left(200m - 0.6c\cdot 1.5\times 10^{-6}s\right)\\ &= \frac{1}{\sqrt{1-(0.6)^2}}\cdot(-70m)\\ &= -87.5m\end{aligned}$$

The answer is negative because event 1 is in front of the train while event 2 is at the rear of the train. This makes $\Delta x = x_2 - x_1$ negative.

See, also, how the time could have been found by taking the point of view of Serge. When the front of the train is besides the first pole, the second post, which is moving at 0.6*c*, must travel 160 m (the distance between the posts according to Serge) to get to the front of the train and then 87.5 m (the length of the train) to arrive besides the back of the train. The distance to travel is, therefore, 247.5 m. The time it takes to travel the distance at 0.6*c* is

$$\Delta t = \frac{247.5m}{0.6c} = 1.375\times 10^{-6}s$$

24. a) According to Sidney, we have

$$\Delta x' = 5ly$$
$$\Delta t' = -8y$$

while we have, according to Gertrude

$$\Delta x = ?$$
$$\Delta t = -7y$$

The interval between the events is

$$\begin{aligned} I' &= (c\Delta t')^2 - (\Delta x')^2 \\ &= (c \cdot -8y)^2 - (5ly)^2 \\ &= 64y^2 \cdot c^2 - 25ly^2 \\ &= 64ly^2 - 25ly^2 \\ &= 39ly^2 \end{aligned}$$

Since the interval is the same for all observers, we have

$$\begin{aligned} I' &= I \\ 39ly^2 &= (c\Delta t)^2 - (\Delta x)^2 \\ 39ly^2 &= (c \cdot -7y)^2 - (\Delta x)^2 \\ 39ly^2 &= 49y^2 \cdot c^2 - (\Delta x)^2 \\ 39ly^2 &= 49ly^2 - (\Delta x)^2 \\ -10ly^2 &= -(\Delta x)^2 \\ \Delta x &= \pm 3.162ly \end{aligned}$$

This, the distance is 3.162 ly.

b) Since we want Δt = -7 y, we must have

$$\begin{aligned} \Delta t &= \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right) \\ -7y &= \gamma\left(-8y + \frac{v \cdot 5y \cdot c}{c^2}\right) \\ -7 &= \gamma\left(-8 + \frac{v}{c} \cdot 5\right) \\ -7 &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(-8 + \frac{v}{c} \cdot 5\right) \end{aligned}$$

Using $v/c = \beta$, the equation becomes

$$-7 = \frac{1}{\sqrt{1-\beta^2}}(-8 + \beta \cdot 5)$$

It only remains to solve for β.

$$-7\sqrt{1-\beta^2} = (-8+\beta\cdot 5)$$
$$49\cdot(1-\beta^2) = (-8+\beta\cdot 5)^2$$
$$49-49\beta^2 = 64-80\beta+25\beta^2$$
$$0 = 15-80\beta+74\beta^2$$
$$\beta = \frac{80\pm\sqrt{80^2-4\cdot 15\cdot 74}}{148}$$
$$\beta = 0.8397 \text{ and } \beta = 0.2414$$

This leads to

$$v = 0.8397c \text{ and } v = 0.2414c$$

These are the two possible answers (With $v = 0.2414c$, we have $\Delta x = 3.162$ ly and with $v = 0.8397c$, we have $\Delta x = -3.162$ ly.)

c) The proper time is found with

$$\begin{aligned}\Delta t_0 &= \tfrac{1}{c}\sqrt{I}\\ &= \tfrac{1}{c}\sqrt{39ly^2}\\ &= \tfrac{1}{c}\sqrt{39y^2\cdot c^2}\\ &= \sqrt{39y^2}\\ &= -6.245y\end{aligned}$$

The negative root was taken since Δt_0 must have the same sign as the Δt used to calculate the interval (since the order in t is the same for all observers if $I > 0$).

25. We will pass from Alice's point of view to Kim's point of view. The speed between these two observers is $v = 0.95c$.

For Aline, the velocity of Mike's spaceship is

$$u_x = -0.95c$$

We're looking for the velocity of Mike's spaceship according to Kim

$$u_x' = ?$$

This velocity is

$$\begin{aligned} u'_x &= \frac{u_x - v}{\left(1 - \frac{vu_x}{c^2}\right)} \\ &= \frac{-0.95c - 0.95c}{\left(1 - \frac{-0.95c \cdot 0.95c}{c^2}\right)} \\ &= \frac{-1.9c}{(1 + 0.95 \cdot 0.95)} \\ &= -0.99869c \end{aligned}$$

26. a)

We will pass from Oscar's point of view to Bertha's point of view. The speed between these two observers is $v = 0.8c$.

For Oscar, the velocity of the missile is

$$u'_x = 0.25c$$

The velocity of the missile according to Bertha is sought

$$u_x = ?$$

This velocity is

$$\begin{aligned} u_x &= \frac{u'_x + v}{\left(1 + \frac{vu'_x}{c^2}\right)} \\ &= \frac{0.25c + 0.8c}{\left(1 + \frac{0.8c \cdot 0.25c}{c^2}\right)} \\ &= \frac{1.05c}{(1 + 0.8 \cdot 0.25)} \\ &= 0.875c \end{aligned}$$

b) As the missile is going slower than Ivan's spaceship, it can't catch him.

27. The length of the train will be found with

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}$$

To calculate the length, we need the length of John's train at rest and the speed of John's train according to Claudette.

First, let's find the length of John's train at rest. Since it is 246 m long when it goes to 0.8*c*, the length at rest is

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}$$

$$246m = L_0\sqrt{1-\frac{(0.8c)^2}{c^2}}$$

$$246m = L_0\sqrt{1-(0.8)^2}$$

$$L_0 = 410m$$

We will now pass from Erin's point of view to Claudette's point of view to find the velocity of John's train according to Claudette. Between these two observers (Erin and Claudette) speed is $v = 0.8c$.

For Erin, the velocity of John's train is

$$u'_x = 0.8c$$

The velocity of John's train according to Claudette is sought.

$$u_x = ?$$

This velocity is

$$\begin{aligned} u_x &= \frac{u'_x + v}{\left(1+\frac{vu'_x}{c^2}\right)} \\ &= \frac{0.8c+0.8c}{\left(1+\frac{0.8c\cdot 0.8c}{c^2}\right)} \\ &= \frac{1.6c}{(1+0.8\cdot 0.8)} \\ &= 0.9756c \end{aligned}$$

Therefore, the length John's train according to Claudette is

$$\begin{aligned} L &= L_0\sqrt{1-\tfrac{v^2}{c^2}} \\ &= 410m\cdot\sqrt{1-\tfrac{(0.9756c)^2}{c^2}} \\ &= 410m\cdot\sqrt{1-(0.9756)^2} \\ &= 90m \end{aligned}$$

28. The Doppler effect will be used. However, to apply this formula, we must be in the reference frame where the observer is at rest. This means that Annabelle's speed according to Esteban must be found from Annabelle's speed according to Lewis.

For Lewis, Annabelle's speed is

$$u_x' = 0.6c$$

Annabelle's speed according to Esteban is sought

$$u_x = ?$$

Therefore

$$\begin{aligned} u_x &= \frac{u_x' + v}{\left(1+\frac{vu_x'}{c^2}\right)} \\ &= \frac{0.6c+0.9c}{\left(1+\frac{0.9c\cdot 0.6c}{c^2}\right)} \\ &= \frac{1.5c}{(1+0.9\cdot 0.6)} \\ &= \frac{75}{77}c \\ &= 0.974c \end{aligned}$$

In Esteban's reference frame, Annabelle's ship approaches at 0.974*c*. As the wave is also heading in the direction of Esteban, Annabelle's speed will be positive in the Doppler effect formula. The received frequency is, therefore,

$$\begin{aligned} f' &= f_0\sqrt{\frac{c+v}{c-v}} \\ &= 100MHz\cdot\sqrt{\frac{c+\frac{75}{77}c}{c-\frac{75}{77}c}} \\ &= 100MHz\cdot\sqrt{\frac{\frac{152}{77}}{\frac{2}{77}}} \\ &= 100MHz\sqrt{\frac{152}{2}} \\ &= 871.8MHz \end{aligned}$$

29. a)

We will pass from Arielle's point of view to Pascal's point of view. The speed between these two observers is $v = 0.8c$.

For Arielle, Pablo's velocity is

$$u_x = 0.95c$$

Pablo's velocity according to Pascal is sought

$$u'_x = ?$$

This velocity is

$$\begin{aligned} u'_x &= \frac{u_x - v}{\left(1-\frac{vu_x}{c^2}\right)} \\ &= \frac{0.95c - 0.8c}{\left(1-\frac{0.8c\cdot 0.95c}{c^2}\right)} \\ &= \frac{0.15c}{(1-0.8\cdot 0.95)} \\ &= 0.625c \end{aligned}$$

b) The two events are:

1) Pablo leaves the Earth.
2) Pablo catches up with Pascal.

Obviously, Pablo is present at both events and so it's him that measures the proper time.

c) According to Ariel, Pablo, travelling at $0.95c$, must catch up with Pascal who is travelling at $0.8c$ while the distance between them is initially 1 ly. Therefore, the time needed to catch up with Pascal is

$$\begin{aligned}\Delta t &= \frac{L}{v_1 - v_2} \\ &= \frac{0.9ly}{0.95c - 0.8c} \\ &= \frac{0,9y \cdot c}{0.15c} \\ &= \frac{0,9y}{0.15} \\ &= 6y\end{aligned}$$

(The formula

$$\Delta t = \frac{L}{v_1 - v_2}$$

was used. This formula, seen in chapter 1 in mechanics, is used to find in how long it will take before two objects that are going at a constant speed meet.)

d) The time according to Pablo can easily be found because we know that Pablo measures the proper time. As the speed of Pablo according to Arielle is $0.95c$, we have

$$\begin{aligned}\Delta t &= \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \\ 6y &= \frac{\Delta t_0}{\sqrt{1 - \frac{(0.95c)^2}{c^2}}} \\ 6y &= \frac{\Delta t_0}{\sqrt{1 - (0.95)^2}} \\ \Delta t_0 &= 1.8735a\end{aligned}$$

Note:

This time could have been found with Lorentz transformations. According to Arielle, the time between the departure of Pablo and the arrival of Pablo is 6 years.

The distance between these events is not 0.9 ly. If Pablo travels for 6 years at 0.95*c*, then he travels 5.7 ly before catching up with Pascal. Therefore, according to Arielle, we have

$$\Delta x = 5.7ly$$
$$\Delta t = 6y$$

The time according to Pablo is thus

$$\begin{aligned}
\Delta t' &= \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \\
&= \frac{1}{\sqrt{1-\frac{(0.95c)^2}{c^2}}}\cdot\left(6y - \frac{0.95c\cdot 5.7ly}{c^2}\right) \\
&= \frac{1}{\sqrt{1-(0.95)^2}}\cdot\left(6y - \frac{0.95c\cdot 5.7y\cdot c}{c^2}\right) \\
&= \frac{1}{\sqrt{1-(0.95)^2}}\cdot\left(6y - 0.95\cdot 5.7y\right) \\
&= \frac{1}{\sqrt{1-(0.95)^2}}\cdot\left(0.585y\right) \\
&= 1.8735y
\end{aligned}$$

e) As the proper time is known, which is the time according to Pablo, the time according to any other observer can be found with the time dilation formula, provided you put the speed of the observer according to the observer who measures the proper time in the time dilation formula. Here, therefore, Pascal's speed according to Pablo must be put in the time dilation formula.

$$\begin{aligned}
\Delta t &= \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}} \\
&= \frac{1.8735a}{\sqrt{1-\frac{(0,625c)^2}{c^2}}} \\
&= \frac{1,8735y}{\sqrt{1-(0.625)^2}} \\
&= 2.4y
\end{aligned}$$

30. a) According to Lou, the time is

$$\begin{aligned}\Delta t &= \frac{L}{v_1 - v_2} \\ &= \frac{12ly}{0.4c - -0.8c} \\ &= \frac{12y \cdot c}{0.4c - -0.8c} \\ &= \frac{12y}{0.4 - -0.8} \\ &= 10y\end{aligned}$$

(The formula

$$\Delta t = \frac{L}{v_1 - v_2}$$

was used. This formula, seen in chapter 1 in mechanics, is used to find in how long it will take before two objects that are going at a constant speed meet.)

b) As the missile travels at $0.8c$, the distance travelled by the missile in 10 years is

$$\begin{aligned}\Delta x &= v\Delta t \\ &= 0.8c \cdot 10y \\ &= 8ly\end{aligned}$$

c) According to Lou, the velocity of the missile is

$$u_x = -0.8c$$

The velocity of the missile according to Paul is sought

$$u'_x = ?$$

This velocity is

$$\begin{aligned} u'_x &= \frac{u_x - v}{\left(1 - \frac{vu_x}{c^2}\right)} \\ &= \frac{-0.8c - 0.4c}{\left(1 - \frac{-0.8c \cdot 0.4c}{c^2}\right)} \\ &= \frac{-1.2c}{(1 - -0.8 \cdot 0.4)} \\ &= -0.9091c \end{aligned}$$

d) The time dilation formula cannot be used directly since neither Lou nor Paul measures the proper time. The Lorentz transformations must then be used. The coordinates of the two events according to Lou must be found, taking the planet as the origin. The two events are:

1) The missile leaves the planet.
2) The missile hit the spaceship.

As the missile travels 8 ly towards the left in 10 years, Lou measures that

$$\begin{aligned} \Delta x &= -8ly \\ \Delta t &= 10y \end{aligned}$$

Therefore, the flight time of the missile according to Paul is

$$\begin{aligned} \Delta t' &= \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \\ &= \frac{1}{\sqrt{1 - \frac{(0.4c)^2}{c^2}}} \cdot \left(10y - \frac{0.4c \cdot (-8ly)}{c^2}\right) \\ &= \frac{1}{\sqrt{1 - (0.4)^2}} \cdot \left(10y - \frac{0.4c \cdot (-8y \cdot c)}{c^2}\right) \\ &= \frac{1}{\sqrt{1 - (0.4)^2}} \cdot (10y + 0.4 \cdot 8y) \\ &= \frac{1}{\sqrt{1 - (0.4)^2}} \cdot (13.2y) \\ &= 14.402y \end{aligned}$$

Note:

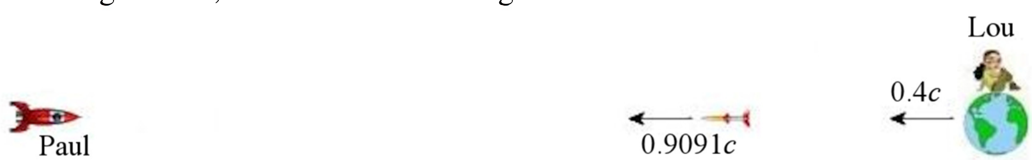
The time can also be found with the time dilation formula by using the time according to the missile (which is the proper time since the missile is present at both events) as an intermediary. First, let's pass from Lou's frame to the missile's frame to find the proper time.

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}$$
$$10y = \frac{\Delta t_0}{\sqrt{1-\frac{(0.8c)^2}{c^2}}}$$
$$10y = \frac{\Delta t_0}{\sqrt{1-(0.8)^2}}$$
$$\Delta t_0 = 6y$$

Then, we pass from the missile's frame to Paul's frame

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}}$$
$$= \frac{6y}{\sqrt{1-\frac{(0.9091c)^2}{c^2}}}$$
$$= \frac{6y}{\sqrt{1-(0.9091)^2}}$$
$$= 14.402y$$

e) According to Paul, we have the following situation.



According to Paul, the missile moves at 0.9091c for 14.402 years. The distance travelled by the missile is thus

$$\Delta x = v\Delta t$$
$$= 0.9091c \cdot 14.402y$$
$$= 13.093ly$$

This gives the distance between Paul and the planet when the missile left the planet. However, while the missile is moving towards Paul, the planet is moving at 0.4*c* towards the spaceship and has travelled

$$\begin{aligned}\Delta x &= v\Delta t \\ &= 0.4c \cdot 14.402y \\ &= 5.761ly\end{aligned}$$

With a planet at 13.093 ly that has moved 5.761 ly, the distance that remains is 13.093 ly – 5.761 ly = 7.332 ly.

Note: Maybe you would have tempted to do this calculation with

$$\begin{aligned}\Delta x' &= \gamma\left(\Delta x - v\Delta t\right) \\ &= \frac{1}{\sqrt{1-\frac{(0.4c)^2}{c^2}}} \cdot \left(\left(-8ly\right) - 0.4c \cdot 10y\right) \\ &= -13.093ly\end{aligned}$$

But this formula gives the distance between the two events a) departure of the missile and b) arrival of the missile. This gives the distance between Paul's spaceship and the position of the planet when the missile left.

31. a) The kinetic energy formula gives

$$E_k = \left(\gamma - 1\right)mc^2$$

$$6.3\times10^{13}J = \left(\gamma-1\right)\cdot 0.145kg \cdot \left(3\times10^8 \tfrac{m}{s}\right)^2$$

$$6.3\times10^{13}J = \left(\gamma-1\right)\cdot 1.302\times10^{16}J$$

$$0.00483 = \gamma - 1$$

$$\gamma = 1.00483$$

$$\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = 1.00483$$

$$\sqrt{1-\tfrac{u^2}{c^2}} = 0.9952$$

$$1-\tfrac{u^2}{c^2} = 0.9904$$

$$\tfrac{u^2}{c^2} = 0.009586$$

$$\tfrac{u}{c} = 0.0979$$

$$u = 0.0979c$$

b) The kinetic energy formula gives

$$\begin{aligned}E_k &= (\gamma - 1)mc^2 \\ &= \left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1\right) \cdot 0.145kg \cdot \left(3\times 10^8 \tfrac{m}{s}\right)^2 \\ &= \left(\frac{1}{\sqrt{1-\frac{(0.95c)^2}{c^2}}} - 1\right) \cdot 1.305\times 10^{16} J \\ &= \left(\frac{1}{\sqrt{1-(0.95)^2}} - 1\right) \cdot 1.305\times 10^{16} J \\ &= 2.874\times 10^{16} J\end{aligned}$$

Now let's calculate how many bomb of Hiroshima this energy represents

$$N = \frac{2.874\times 10^{16} J}{6.3\times 10^{13} J} = 456.2$$

32. a) The momentum is

$$\begin{aligned}p &= \gamma mu \\ &= \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} mu \\ &= \frac{1}{\sqrt{1-\frac{(0.95c)^2}{c^2}}} \cdot 1.673\times 10^{-27} kg \cdot 0.95c \\ &= \frac{1}{\sqrt{1-(0.95)^2}} \cdot 1.673\times 10^{-27} kg \cdot 0.95 \cdot 3\times 10^8 \tfrac{m}{s} \\ &= 1.527\times 10^{-18} \tfrac{kgm}{s}\end{aligned}$$

b) The kinetic energy is

$$\begin{aligned}
E_k &= (\gamma - 1) mc^2 \\
&= \left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1 \right) \cdot 1.673\times 10^{-27} kg \cdot \left(3\times 10^8 \tfrac{m}{s}\right)^2 \\
&= \left(\frac{1}{\sqrt{1-\frac{(0.95c)^2}{c^2}}} - 1 \right) \cdot 1.5057\times 10^{-10} J \\
&= \left(\frac{1}{\sqrt{1-(0.95)^2}} - 1 \right) \cdot 1.5057\times 10^{-10} J \\
&= 3.3164\times 10^{-10} J \\
&= 2070 MeV
\end{aligned}$$

c) The relativistic energy is

$$\begin{aligned}
E &= \gamma mc^2 \\
&= \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \cdot 1.673\times 10^{-27} kg \cdot \left(3\times 10^8 \tfrac{m}{s}\right)^2 \\
&= \frac{1}{\sqrt{1-\frac{(0.95c)^2}{c^2}}} \cdot 1.5057\times 10^{-10} J \\
&= \frac{1}{\sqrt{1-(0.95)^2}} \cdot 1.5057\times 10^{-10} J \\
&= 4.822\times 10^{-10} J \\
&= 3010 MeV
\end{aligned}$$

d) The mass is

$$\begin{aligned}
m_{tot} &= m + m_{\text{Kinetic energy}} \\
&= m + \frac{E_{\text{Kinetic energy}}}{c^2} \\
&= 1.673\times 10^{-27} kg + \frac{3.3164\times 10^{-10} J}{c^2} \\
&= 1.673\times 10^{-27} kg + 3.684\times 10^{-27} kg \\
&= 5.358\times 10^{-27} kg
\end{aligned}$$

Or, we could have used the fact that $m_{tot} = \gamma m$ (as shown in the notes) and obtain

$$\begin{aligned} m_{tot} &= \gamma m \\ &= \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \cdot 1.673\times10^{-27} kg \\ &= \frac{1}{\sqrt{1-\frac{(0.95c)^2}{c^2}}} \cdot 1.673\times10^{-27} kg \\ &= \frac{1}{\sqrt{1-(0.95)^2}} \cdot 1.673\times10^{-27} kg \\ &= 5.358\times10^{-27} kg \end{aligned}$$

33. a) At 0.7*c*, the relativistic energy is

$$\begin{aligned} E_k &= \gamma mc^2 \\ &= \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \cdot 9.11\times10^{-31} kg \cdot \left(3\times10^8 \tfrac{m}{s}\right)^2 \\ &= \frac{1}{\sqrt{1-\frac{(0.7c)^2}{c^2}}} \cdot 8.199\times10^{-14} J \\ &= \frac{1}{\sqrt{1-(0.7)^2}} \cdot 8.199\times10^{-14} J \\ &= 1.148\times10^{-13} J \\ &= 716.7 keV \end{aligned}$$

At 0.8*c*, the relativistic energy is

$$\begin{aligned} E_k &= \gamma mc^2 \\ &= \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \cdot 9.11\times10^{-31} kg \cdot \left(3\times10^8 \tfrac{m}{s}\right)^2 \\ &= \frac{1}{\sqrt{1-\frac{(0.8c)^2}{c^2}}} \cdot 8.199\times10^{-14} J \\ &= \frac{1}{\sqrt{1-(0.8)^2}} \cdot 8.199\times10^{-14} J \\ &= 1.3665\times10^{-13} J \\ &= 853.0 keV \end{aligned}$$

The difference in energy is the energy that must be given to the electron. This difference is

$$853.0keV - 716.7keV = 136.3keV$$

Note: The kinetic energy could also have been instead of the relativistic energy.

b) At 0.8c, the relativistic energy is

$$\begin{aligned} E_k &= \gamma mc^2 \\ &= \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \cdot 9.11\times10^{-31}kg \cdot \left(3\times10^8 \tfrac{m}{s}\right)^2 \\ &= \frac{1}{\sqrt{1-\frac{(0.8c)^2}{c^2}}} \cdot 8.199\times10^{-14}J \\ &= \frac{1}{\sqrt{1-(0.8)^2}} \cdot 8.199\times10^{-14}J \\ &= 1.3665\times10^{-13}J \\ &= 853.0keV \end{aligned}$$

At 0.9c, the relativistic energy is

$$\begin{aligned} E_k &= \gamma mc^2 \\ &= \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \cdot 9.11\times10^{-31}kg \cdot \left(3\times10^8 \tfrac{m}{s}\right)^2 \\ &= \frac{1}{\sqrt{1-\frac{(0.9c)^2}{c^2}}} \cdot 8.199\times10^{-14}J \\ &= \frac{1}{\sqrt{1-(0.9)^2}} \cdot 8.199\times10^{-14}J \\ &= 1.881\times10^{-13}J \\ &= 1174.1keV \end{aligned}$$

The difference in energy is the energy that must be given to the electron. This difference is

$$1174.1keV - 853.0keV = 321.1keV$$

34. The mass is found with

$$E = mc^2$$
$$3.83\times10^{26}J = m\cdot\left(3\times10^8 \tfrac{m}{s}\right)^2$$
$$m = 4.2556\times10^9 kg$$

The Sun, therefore, loses 4.26 million tons each second.

35. Of course, the speed of the proton can be found and then momentum can be calculated but the answer can be found more quickly using

$$E^2 - (pc)^2 = \left(mc^2\right)^2$$

This equation gives

$$\left(50\times10^9\cdot1.602\times10^{-19}J\right)^2 - (pc)^2 = \left(1.673\times10^{-27}kg\cdot c^2\right)^2$$
$$\left(8.01\times10^{-9}J\right)^2 - (pc)^2 = \left(1.5057\times10^{-10}J\right)^2$$
$$\left(8.01\times10^{-9}J\right)^2 - \left(1.5057\times10^{-10}J\right)^2 = (pc)^2$$
$$pc = 8.0086\times10^{-9}J$$
$$p = 2.6695\times10^{-17}\tfrac{kg\,m}{s}$$

36. a) The value of γ for this electron is found with

$$E_k = (\gamma-1)mc^2$$
$$10^{12}\cdot1.602\times10^{-19}J = (\gamma-1)\cdot9.11\times10^{-31}kg\cdot\left(3\times10^8\tfrac{m}{s}\right)^2$$
$$\gamma-1 = 1\,953\,897$$
$$\gamma = 1\,953\,898$$

Thus, the length of the tunnel according to the electron is

$$\begin{aligned} L &= L_0\sqrt{1-\tfrac{v^2}{c^2}} \\ &= \frac{L_0}{\gamma} \\ &= \frac{10000m}{1\,953\,898} \\ &= 5.118mm \end{aligned}$$

b) According to observers on Earth, the speed of the electron is

$$\begin{aligned} \gamma &= 1\,953\,898 \\ \frac{1}{\sqrt{1-\tfrac{v^2}{c^2}}} &= 1\,953\,898 \\ \sqrt{1-\tfrac{v^2}{c^2}} &= 5.118\times10^{-7} \\ 1-\tfrac{v^2}{c^2} &= 2.619\times10^{-13} \\ 1-2.619\times10^{-13} &= \tfrac{v^2}{c^2} \\ 1 &\approx \tfrac{v^2}{c^2} \\ v &\approx c \end{aligned}$$

These electrons therefore almost travel at the speed of light (in fact 0.999 999 999 999 87*c*). The time to move from one end of the tunnel to the other is thus

$$\Delta t = \frac{10000m}{c} = 3.333\times10^{-5}s$$

c) In the reference frame of the electron, the tunnel is 5.118mm long and is moving almost at the speed of light. To move from one end to the other, the time is

$$\Delta t_0 = \frac{0.005118m}{c} = 1.706\times10^{-11}s$$

37. a)

In an inelastic collision, the momentum is conserved. Before the collision, the momentum is

$$\begin{aligned} p_{before} &= p_{ball} + p_{liner} \\ &= \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} mu + 0 \\ &= \frac{1}{\sqrt{1-\left(0.5^2\right)}} \cdot 5kg \cdot 0.5c \\ &= 8.66\times 10^8 \tfrac{kgm}{s} \end{aligned}$$

After the collision, the momentum is

$$\begin{aligned} p_{after} &= p_{ball+liner} \\ &= m_{ball+liner} u \\ &= 75{,}000{,}005kg \cdot u \end{aligned}$$

The relativistic formula was not used because the speed will be much smaller than the speed of light. (In fact, the speed is not known beforehand but we take the simplest formula, i.e., the non-relativistic formula, to find the speed. If we then realize that the speed is too high, the solution is made again with the relativistic formula.)

Conservation of momentum gives

$$\begin{aligned} p_{before} &= p_{after} \\ 8.66\times 10^8 \tfrac{kgm}{s} &= 75{,}000{,}005kg \cdot u \\ u &= 11.547 \tfrac{m}{s} \end{aligned}$$

b) The initial kinetic energy is

$$\begin{aligned} E_{k\ before} &= E_{k\ ball} + E_{k\ liner} \\ &= \left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1 \right) mc^2 + 0 \\ &= \left(\frac{1}{\sqrt{1-\left(0.5^2\right)}} - 1 \right) \cdot 5kg \cdot \left(3\times 10^8 \tfrac{m}{s}\right)^2 \\ &= 6.962\times 10^{16} J \end{aligned}$$

After the collision, the kinetic energy is

$$\begin{aligned} E_{k\ after} &= E_{k\ ball+liner} \\ &= \frac{1}{2} m_{ball+liner} u^2 \\ &= \frac{1}{2} \cdot 75{,}000{,}005 kg \cdot \left(11.547 \tfrac{m}{s}\right)^2 \\ &= 5 \times 10^9 J \end{aligned}$$

The relativistic formula was not used because the speed is much smaller than the speed of light.

The energy lost in the collision is thus

$$\begin{aligned} \Delta E_k &= E_{k\ after} - E_{k\ before} \\ &= 5 \times 10^9 J - 6.962 \times 10^{16} J \\ &= -6.962 \times 10^{16} J \end{aligned}$$

c) The number of bombs of Hiroshima is

$$N = \frac{6.962 \times 10^{16} J}{6.3 \times 10^{13} J} = 1105$$

Probably, not much of the liner is left…

38. a)

In an inelastic collision, the momentum is conserved. Before the collision, the momentum is

$$\begin{aligned} p_{before} &= p_{neutron} + p_{nucleus} \\ &= \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} mu + 0 \\ &= \frac{1}{\sqrt{1 - (0.8^2)}} 1.675 \times 10^{-27} kg \cdot 0.8c \\ &= \frac{1}{\sqrt{1 - (0.8^2)}} 1.675 \times 10^{-27} kg \cdot 0.8 \cdot 3 \times 10^8 \tfrac{m}{s} \\ &= 6.7 \times 10^{-19} \tfrac{kgm}{s} \end{aligned}$$

After the collision, the momentum is

$$\begin{aligned} p_{after} &= p_{nucleus+neutron} \\ &= \gamma m_{nucleus+neutron} u \\ &= 8.323\times 10^{-27} kg \cdot \gamma u \end{aligned}$$

Conservation of momentum then gives

$$\begin{aligned} p_{before} &= p_{after} \\ 6.7\times 10^{-19} \tfrac{kgm}{s} &= 8.323\times 10^{-27} kg \cdot \gamma u \\ \gamma u &= 8.05\times 10^{7} \tfrac{m}{s} \end{aligned}$$

It simply remains to solve this equation for *u.*

$$8.05\times 10^{7} \tfrac{m}{s} = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}} u$$

$$8.05\times 10^{7} \tfrac{m}{s} \cdot \sqrt{1-\left(\tfrac{u}{c}\right)^2} = u$$

$$\left(8.05\times 10^{7} \tfrac{m}{s}\right)^2 \left(1-\left(\tfrac{u}{c}\right)^2\right) = u^2$$

$$\left(8.05\times 10^{7} \tfrac{m}{s}\right)^2 - \frac{\left(8.05\times 10^{7} \frac{m}{s}\right)^2}{c^2} \cdot u^2 = u^2$$

$$\left(8.05\times 10^{7} \tfrac{m}{s}\right)^2 = u^2 + \frac{\left(8.05\times 10^{7} \frac{m}{s}\right)^2}{c^2} \cdot u^2$$

$$\left(8.05\times 10^{7} \tfrac{m}{s}\right)^2 = \left(1 + \frac{\left(8.05\times 10^{7} \frac{m}{s}\right)^2}{c^2}\right) \cdot u^2$$

$$u^2 = \frac{\left(8.05\times 10^{7} \frac{m}{s}\right)^2}{1 + \frac{\left(8.05\times 10^{7} \frac{m}{s}\right)^2}{c^2}}$$

$$u = \frac{8.05\times 10^{7} \frac{m}{s}}{\sqrt{1 + \frac{\left(8.05\times 10^{7} \frac{m}{s}\right)^2}{c^2}}}$$

$$u = 7.775\times 10^{7} \tfrac{m}{s}$$

$$u = 0.2592c$$

Alternate way to obtain u

As the momentum after the collision is 6.7 x 10^{-19} kgm/s, we have

$$E^2 - p^2c^2 = m^2c^4$$
$$E^2 - \left(6.7\times10^{-19}\,\tfrac{kgm}{s}\right)^2 \cdot \left(3\times10^8\,\tfrac{m}{s}\right)^2 = \left(8.323\times10^{-27}\,kg\right)^2 \cdot \left(3\times10^8\,\tfrac{m}{s}\right)^4$$
$$E = 7.7557\times10^{-10}\,J$$

Then, we have

$$E = \gamma mc^2$$
$$7.7557\times10^{-10}\,J = \gamma \cdot 8.323\times10^{-27}\,kg \cdot \left(3\times10^8\,\tfrac{m}{s}\right)^2$$
$$\gamma = 1.03538$$

Therefore

$$\gamma = 1.03538$$
$$\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = 1.03538$$
$$u = 0.2592c$$

b) The initial kinetic energy is

$$\begin{aligned}E_{k\,before} &= E_{k\,neutron} + E_{k\,nucleus}\\ &= \left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1\right)mc^2 + 0\\ &= \left(\frac{1}{\sqrt{1-\left(0.8^2\right)}} - 1\right)\cdot 1.675\times10^{-27}\,kg \cdot \left(3\times10^8\,\tfrac{m}{s}\right)^2\\ &= 1.005\times10^{-10}\,J\end{aligned}$$

After the collision, the kinetic energy is

$$\begin{aligned} E_{k\,after} &= E_{k\,neutron+nucleus} \\ &= \left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1 \right) mc^2 \\ &= \left(\frac{1}{\sqrt{1-\left(0.2592^2\right)}} - 1 \right) \cdot 8.323 \times 10^{-27} kg \cdot \left(3 \times 10^8 \tfrac{m}{s} \right)^2 \\ &= 2.65 \times 10^{-11} J \end{aligned}$$

The energy lost in the collision is thus

$$\begin{aligned} \Delta E_k &= E_{k\,after} - E_{k\,before} \\ &= 2.65 \times 10^{-11} J - 1.005 \times 10^{-10} J \\ &= -7.4 \times 10^{-11} J \\ &= -462 MeV \end{aligned}$$

(This is a lot for such a small nucleus. It's actually more than 15 times the binding energy of the nucleus. It is probable that this nucleus would be split into protons and neutrons.)

39. a)

As in any process, the relativistic energy is conserved. Since the mass energy is 139.6 MeV before the disintegration and 105.7 MeV after the disintegration, the kinetic energy of the two particles after decay must be of 33.9 MeV.

Since the momentum is zero before the disintegration, it must also be zero after the disintegration. So, we have the following equations.

$$\begin{gathered} E_{k1} + E_{k2} = 33.9 MeV \\ p_1 + p_2 = 0 \end{gathered}$$

This last equation gives

$$\begin{gathered} p_1 = -p_2 \\ p_1^2 = p_2^2 \\ p_1^2 c^2 = p_2^2 c^2 \\ E_1^2 - m_1^2 c^4 = E_2^2 \end{gathered}$$

Since

$$E_1 + E_2 = 139.6MeV$$

The equation becomes

$$E_1^2 - m_1^2c^4 = \left(139.6MeV - E_1\right)^2$$
$$E_1^2 - m_1^2c^4 = \left(139.6MeV\right)^2 - 279.2MeV \cdot E_1 + E_1^2$$
$$-m_1^2c^4 = \left(139.6MeV\right)^2 - 279.2MeV \cdot E_1$$

Solving for the energy of particle 1, the equation becomes

$$E_1 = \frac{\left(139.6MeV\right)^2 + m_1^2c^4}{279.2MeV}$$

With the values of the mass, we get

$$\begin{aligned} E_1 &= \frac{\left(139.6MeV\right)^2 + \left(105.7MeV\right)^2}{279.2MeV} \\ &= 109.8MeV \end{aligned}$$

With this value, we can find easily that

$$\begin{aligned} E_2 &= 139.6MeV - E_1 \\ &= 29.8MeV \end{aligned}$$

As the relativistic energy is the sum of the mass energy and of the kinetic energy, the kinetic energy of particle 1 (muon) is 4.1 MeV and of particle 2 (neutrino) is 29.8 MeV. The sum of these energies is indeed 33.9 MeV as predicted.

b) It's easy enough to find the speed of the neutrino. Because we assume that it has no mass, then its speed must be equal to the speed of light.

For the muon, the value of γ can be found with

$$E_1 = \gamma_1 m_1 c^2$$
$$109.8MeV = \gamma_1 \cdot 105.7MeV$$
$$\gamma_1 = 1.0389$$

With γ, the speed can be found.

$$1.0389 = \frac{1}{\sqrt{1-\left(\frac{u_1}{c}\right)^2}}$$

$$u_1 = 0.2712c$$

40. First Version (the simplest)

We start with

$$E^2 - \left(mc^2\right)^2 = \left(pc\right)^2$$

Since $x^2 - 1 = (x + 1)(x - 1)$, we have

$$E^2 - \left(mc^2\right)^2 = \left(pc\right)^2$$

$$\left(E + mc^2\right)\left(E - mc^2\right) = \left(pc\right)^2$$

Since $E - mc^2$ is the kinetic energy, we obtain

$$\left(E + mc^2\right)E_k = \left(pc\right)^2$$

As $E = \gamma mc^2$, we arrive at

$$\left(E + mc^2\right)E_k = \left(pc\right)^2$$

$$\left(\gamma mc^2 + mc^2\right)E_k = \left(pc\right)^2$$

$$\left(\gamma + 1\right)mc^2 E_k = p^2c^2$$

$$\left(\gamma + 1\right)mE_k = p^2$$

$$E_k = \frac{p^2}{\left(\gamma + 1\right)m}$$

Second Version

It is known that

$$E_k = \left(\gamma - 1\right)mc^2 \qquad p = \gamma mv$$

We can then write

$$\begin{aligned} E_k &= (\gamma - 1) mc^2 \\ &= (\gamma - 1) mc^2 \frac{p^2}{p^2} \\ &= (\gamma - 1) mc^2 \frac{p^2}{\gamma^2 m^2 v^2} \\ &= \frac{\gamma - 1}{\gamma^2} \frac{c^2}{v^2} \frac{p^2}{m} \end{aligned}$$

However, we have

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}$$

$$\frac{1}{\gamma^2} = 1 - \frac{v^2}{c^2}$$

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

$$\frac{v^2}{c^2} = \frac{\gamma^2}{\gamma^2} - \frac{1}{\gamma^2}$$

$$\frac{v^2}{c^2} = \frac{\gamma^2 - 1}{\gamma^2}$$

$$\frac{c^2}{v^2} = \frac{\gamma^2}{\gamma^2 - 1}$$

If we use this equation in

$$E_k = \frac{\gamma - 1}{\gamma^2} \frac{c^2}{v^2} \frac{p^2}{m}$$

We arrive at

$$E_k = \frac{\gamma - 1}{\gamma^2} \frac{\gamma^2}{\gamma^2 - 1} \frac{p^2}{m}$$

This gives

$$\begin{aligned}E_k &= \frac{\gamma-1}{\gamma^2-1}\frac{p^2}{m}\\&= \frac{\gamma-1}{(\gamma-1)(\gamma+1)}\frac{p^2}{m}\\&= \frac{1}{\gamma+1}\frac{p^2}{m}\end{aligned}$$

41. Let's start with

$$\frac{1}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\cdot\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\left(1-\frac{vu}{c^2}\right)$$

Since

$$u' = \frac{u-v}{1-\frac{uv}{c^2}}$$

we have

$$1-\tfrac{u'^2}{c^2}=1-\left(\frac{u-v}{1-\frac{uv}{c^2}}\right)^2\cdot\frac{1}{c^2}$$

$$=1-\left(\frac{\frac{u}{c}-\frac{v}{c}}{1-\frac{uv}{c^2}}\right)^2$$

$$=1-\frac{\frac{u^2}{c^2}-2\frac{uv}{c^2}+\frac{v^2}{c^2}}{\left(1-\frac{uv}{c^2}\right)^2}$$

$$=\frac{\left(1-\frac{uv}{c^2}\right)^2}{\left(1-\frac{uv}{c^2}\right)^2}-\frac{\frac{u^2}{c^2}-2\frac{uv}{c^2}+\frac{v^2}{c^2}}{\left(1-\frac{uv}{c^2}\right)^2}$$

$$=\frac{1-2\frac{uv}{c^2}+\frac{u^2}{c^2}\frac{v^2}{c^2}}{\left(1-\frac{uv}{c^2}\right)^2}-\frac{\frac{u^2}{c^2}-2\frac{uv}{c^2}+\frac{v^2}{c^2}}{\left(1-\frac{uv}{c^2}\right)^2}$$

$$=\frac{1-\frac{u^2}{c^2}-\frac{v^2}{c^2}+\frac{u^2}{c^2}\frac{v^2}{c^2}}{\left(1-\frac{uv}{c^2}\right)^2}$$

$$=\frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{uv}{c^2}\right)^2}$$

Thus

$$\sqrt{1-\tfrac{u'^2}{c^2}}=\frac{\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}}{\left(1-\frac{uv}{c^2}\right)}$$

$$\frac{1}{\sqrt{1-\frac{u'^2}{c^2}}}=\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\cdot\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(1-\frac{uv}{c^2}\right)$$

There it is, the proof is done.

The other formula is

$$\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}=\frac{1}{\sqrt{1-\frac{u'^2}{c^2}}}\cdot\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\left(1+\frac{vu'}{c^2}\right)$$

Since

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

we have

$$\begin{aligned}
1 - \frac{u^2}{c^2} &= 1 - \left(\frac{u' + v}{1 + \frac{u'v}{c^2}} \right)^2 \cdot \frac{1}{c^2} \\
&= 1 - \left(\frac{\frac{u'}{c} + \frac{v}{c}}{1 + \frac{u'v}{c^2}} \right)^2 \\
&= 1 - \frac{\frac{u'^2}{c^2} + 2\frac{u'v}{c^2} + \frac{v^2}{c^2}}{\left(1 + \frac{u'v}{c^2}\right)^2} \\
&= \frac{\left(1 + \frac{u'v}{c^2}\right)^2}{\left(1 + \frac{u'v}{c^2}\right)^2} - \frac{\frac{u'^2}{c^2} + 2\frac{u'v}{c^2} + \frac{v^2}{c^2}}{\left(1 + \frac{u'v}{c^2}\right)^2} \\
&= \frac{1 + 2\frac{u'v}{c^2} + \frac{u'^2}{c^2}\frac{v^2}{c^2}}{\left(1 + \frac{u'v}{c^2}\right)^2} - \frac{\frac{u'^2}{c^2} + 2\frac{u'v}{c^2} + \frac{v^2}{c^2}}{\left(1 + \frac{u'v}{c^2}\right)^2} \\
&= \frac{1 - \frac{u'^2}{c^2} - \frac{v^2}{c^2} + \frac{u'^2}{c^2}\frac{v^2}{c^2}}{\left(1 + \frac{u'v}{c^2}\right)^2} \\
&= \frac{\left(1 - \frac{u'^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{u'v}{c^2}\right)^2}
\end{aligned}$$

Thus

$$\sqrt{1 - \frac{u^2}{c^2}} = \frac{\sqrt{1 - \frac{u'^2}{c^2}}\sqrt{1 - \frac{v^2}{c^2}}}{\left(1 + \frac{u'v}{c^2}\right)}$$

$$\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{u'^2}{c^2}}} \cdot \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(1 + \frac{u'v}{c^2}\right)$$

42. If an object is moving at speed u', the x-component of its momentum is

$$p_x' = \frac{1}{\sqrt{1 - \frac{u'^2}{c^2}}} m u_x'$$

Transforming to another observer who sees the object moving at speed u, the result is

$$\begin{aligned}
p'_x &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\left(1-\frac{uv}{c^2}\right)m\frac{u-v}{1-\frac{uv}{c^2}} \\
&= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}m(u-v) \\
&= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}mu-\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}mv\right) \\
&= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}mu-\frac{v}{c^2}\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}mc^2\right)
\end{aligned}$$

Since

$$p_x = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}mu \qquad \text{and} \qquad E = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}mc^2$$

the momentum is

$$p'_x = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(p_x-\frac{v}{c^2}E\right)$$

If an object is moving at speed u', its relativistic energy is

$$E' = \frac{1}{\sqrt{1-\frac{u'^2}{c^2}}}mc^2$$

Transforming to another observer who sees the object moving at speed u, the result is

$$\begin{aligned}
E' &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\left(1-\frac{uv}{c^2}\right)mc^2 \\
&= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}mc^2-\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}muv\right)
\end{aligned}$$

Since

$$p_x = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}mu \qquad \text{and} \qquad E = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}mc^2$$

the energy is

$$E' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(E - vp_x\right)$$

43. a)

For Felix, the extremities of the moving walkway are moving. This means that the walkway is contracted according to Felix. Thus, the length of the walkway according to Felix is

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}$$

The distance between the moving people is s according to Beatrix. This is a contracted distance. According to Felix, the people are not moving and so the distance between people is equal to the non-contracted distance s_0.

$$s_0 = \frac{s}{\sqrt{1-\frac{v^2}{c^2}}}$$

Thus, the length of the walkway is L and the distance between the people is s_0. Therefore, the number of people on this side

$$\begin{aligned} N_1 &= \frac{L}{s_0} \\ &= \frac{L_0\sqrt{1-\frac{v^2}{c^2}}}{s/\sqrt{1-\frac{v^2}{c^2}}} \\ &= \frac{L_0}{s}\left(1-\frac{v^2}{c^2}\right) \end{aligned}$$

As the number of people according to Beatrix is

$$N = \frac{2L_0}{s}$$

we obtain

$$N_1 = \frac{N}{2}\left(1 - \frac{v^2}{c^2}\right)$$

b) According to Felix, the people on the other side of the walkway are moving at the speed

$$u' = \frac{v - -v}{1 - \frac{v(-v)}{c^2}}$$

This means that the distance between the people of this side is strongly contracted. The distance between the people is

$$s' = s_0 \sqrt{1 - \frac{u'^2}{c^2}}$$

Knowing that the speed u' is the result of the combination of speed v and speed –v, we have, using the result of the first challenge,

$$\begin{aligned} s' &= s_0 \sqrt{1 - \frac{u'^2}{c^2}} \\ &= \frac{s_0 \sqrt{1 - \frac{v^2}{c^2}} \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v(-v)}{c^2}} \\ &= \frac{s_0 \left(1 - \frac{v^2}{c^2}\right)}{1 + \frac{v^2}{c^2}} \end{aligned}$$

Since

$$s_0 = \frac{s}{\sqrt{1 - \frac{v^2}{c^2}}}$$

the distance becomes

$$\begin{aligned} s' &= \frac{s}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{\left(1 - \frac{v^2}{c^2}\right)}{1 + \frac{v^2}{c^2}} \\ &= \frac{s\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v^2}{c^2}} \end{aligned}$$

Thus, the length of the walkway is L and the distance between the people is s'. Therefore, the number of people on this side

$$\begin{aligned}N_2 &= \frac{L}{s'} \\ &= \frac{L_0\sqrt{1-\frac{v^2}{c^2}}}{s\sqrt{1-\frac{v^2}{c^2}}\,/\left(1+\frac{v^2}{c^2}\right)} \\ &= \frac{L_0}{s}\left(1+\frac{v^2}{c^2}\right)\end{aligned}$$

As the number of people according to Beatrix is

$$N = \frac{2L_0}{s}$$

we obtain

$$N_2 = \frac{N}{2}\left(1+\frac{v^2}{c^2}\right)$$

c) The total number of people is

$$\begin{aligned}N &= N_1 + N_2 \\ &= \frac{N}{2}\left(1-\frac{v^2}{c^2}\right)+\frac{N}{2}\left(1+\frac{v^2}{c^2}\right) \\ &= N\end{aligned}$$

(It makes sense that both count the same number of people. People cannot disappear depending on the speed of an observer...)