5 SUPERPOSITION OF WAVES IN 1D

The largest string of a guitar has a length of 92.9 cm and a mass of 5.58 g. Once on the guitar, there is 65.5 cm between the two points of attachment of the string. What should be the tension of the string so that the frequency of the first harmonic is 82.4 Hz (which is the frequency that the largest string of a guitar should have)?

mathforum.org/mathimages/index.php/Special:Thumb

Discover how to solve this problem in this chapter.

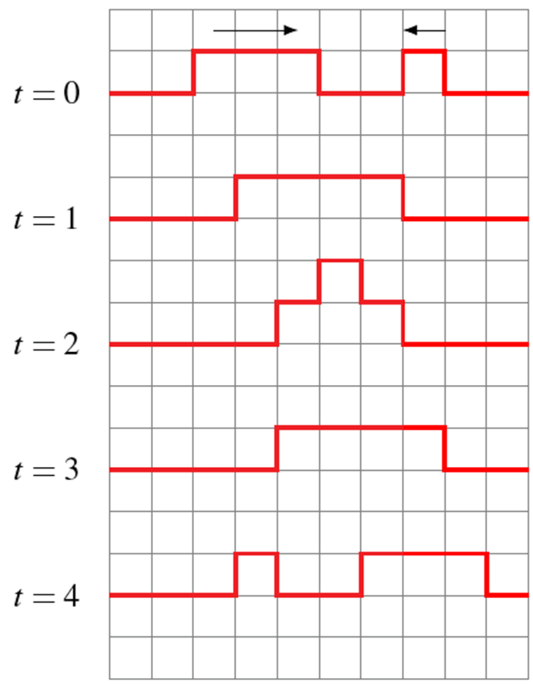
5.1 SUPERPOSITION PRINCIPLE

There is superposition when several waves meet at the same place in a medium. The result of the encounter of two waves is particularly simple: the displacements caused by each wave add up. This is called the superposition principle.

Superposition Principle

$$y_{tot} = y_1 + y_2 + y_3 + \dots$$

Let's illustrate this principle with an example. Two waves with a rectangular shape are heading towards each other. The distances indicated by a grid in the diagram are in centimetres. Each wave has a speed of 1 cm/s and causes the rope to shift 1 cm from the equilibrium position.

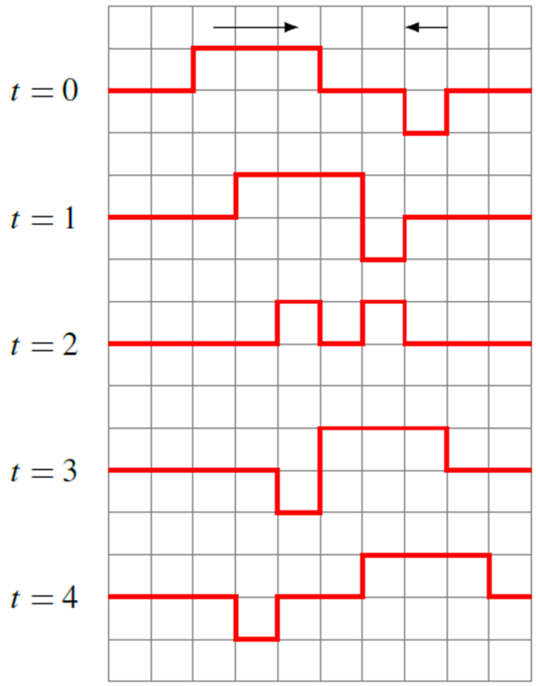


At t = 1 s, the fronts of each wave come into contact and the interference begins. At t = 2 s, the 1 cm long wave is found in the middle of the 3 cm long wave. There, the displacements caused by each wave (which is 1 cm) add up to obtain a displacement of 2 cm. At t = 3 s, the interference ends and at t = 4 s, the two original waves are found again, and each is continuing on its way. The passage of the two waves one through the other does not affect at all the shape of the waves. They are back to their original shape once they have interfered.

In this case, the displacement of the rope is greater at the places where the waves overlap than the displacement caused by a single wave. When this happens, **constructive interference** is obtained.

This beautiful clip illustrates the result of two waves overlapping on a string. The top part of the clip shows the wave going towards the right. The middle part shows the wave going towards the left and at the bottom part of the clip shows the addition of these two waves.
http://www.youtube.com/watch?v=8IRZYOC7DeU

You can also watch this demonstration.
http://www.youtube.com/watch?v=YviTr5tH8jw



In this other example, one of the waves generates a negative displacement on the rope. At t = 2 s, the short wave is located in the middle of the long wave. At the places where the waves are superimposed, the displacement made by the long wave (1 cm) is added to the displacement made by the short wave (-1 cm). The result is 0 cm and the rope is at its equilibrium position everywhere where the short wave interferes with the longer wave.

In this case, the displacement of the rope is zero where the waves overlap. When the resulting displacement on the two waves is smaller than the displacement made by only one wave (like here), **destructive interference** is obtained.

It can still be seen that, after the passage of the waves one through the other, each wave retakes exactly the same shape it had before the encounter of the waves.

Again, the interference of two waves can be seen in this clip but, this time, a wave generates a positive displacement (the one moving towards the right) while the other generates a negative displacement (the one moving towards the left).
http://www.youtube.com/watch?v=95macpu6xgM
Here's a demonstration of destructive interference.
http://www.youtube.com/watch?v=URRe-hOKuMs

These elements are only true for waves travelling in a linear medium. In a non-linear medium, the wave resulting from the superposition of waves is not simply the sum of each wave. Also, in a non-linear medium, the shape of a wave after its passage through another wave is different from the shape it had before the encounter. Non-linear media will not be considered in these notes.

5.2 SUPERPOSITION OF TWO SINE WAVES

Resulting Amplitude for Two Sine Waves With the Same Amplitude

To begin with, two sine waves that have the same frequency and that are travelling in the same direction will be superimposed.

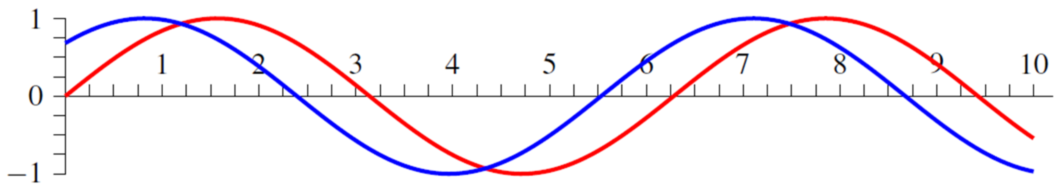
Wave 1 is

$$y_1 = A\sin\left(kx - \omega t + \phi_1\right)$$

Wave 2 is

$$y_2 = A\sin\left(kx - \omega t + \phi_2\right)$$

The amplitude, the wavelength, and the frequency are the same, but the two waves can have different phase constant, as these 2 waves in the diagram.



The resulting wave is the result of the superimposition of these two waves, which is

$$y_{tot} = A\sin\left(kx - \omega t + \phi_1\right) + A\sin\left(kx - \omega t + \phi_2\right)$$
$$y_{tot} = A\left(\sin\left(kx - \omega t + \phi_1\right) + \sin\left(kx - \omega t + \phi_2\right)\right)$$

This addition can be performed by using the identity

$$\sin X + \sin Y = 2\cos\left(\frac{X-Y}{2}\right)\sin\left(\frac{X+Y}{2}\right)$$

The result of the addition is thus

$$y_{tot} = \underbrace{2A\cos\left(\frac{\phi_1 - \phi_2}{2}\right)}_{\text{Amplitude}}\underbrace{\sin\left(kx - \omega t + \frac{\phi_1 + \phi_2}{2}\right)}_{\text{Oscillation as a function of time}}$$

The interesting result here is the amplitude of the resultant wave. This amplitude will be denoted A_{tot}. (The amplitude of each wave is denoted A, while the resulting amplitude of two superimposed waves is denoted A_{tot}.)

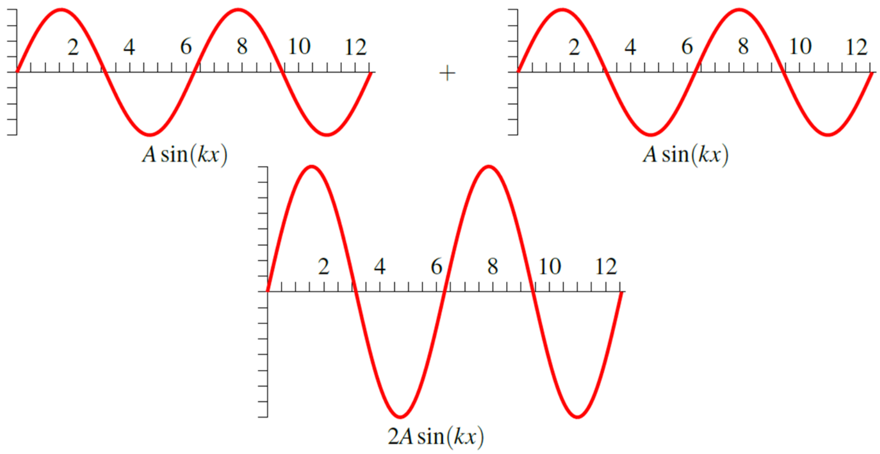
Resulting Amplitude of Two Interfering Waves With the Same Amplitude *A*

$$A_{tot} = \left|2A\cos\left(\frac{\Delta\phi}{2}\right)\right|$$

The absolute value of the result is taken since a negative sign in front of an amplitude is just a hidden π radians phase shift that does not change the amplitude. $\Delta\phi$ is called the *phase difference*.

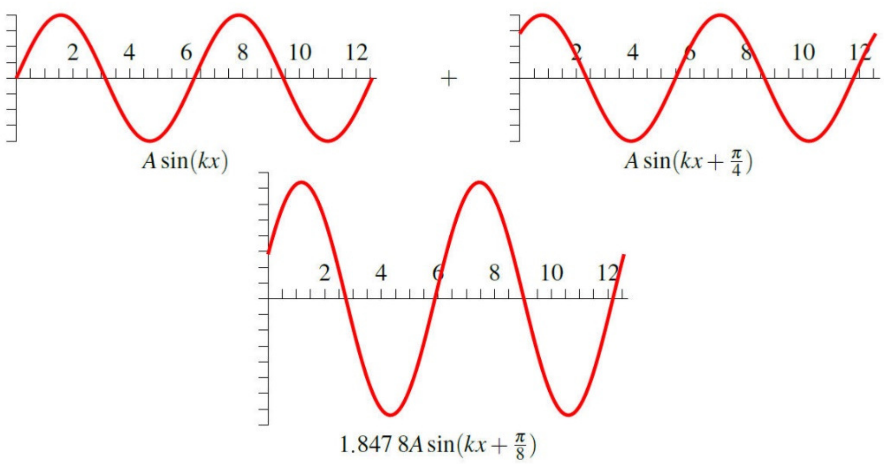
Let's have a look at some cases.

For two waves in phase ($\Delta\phi = 0$), the result is

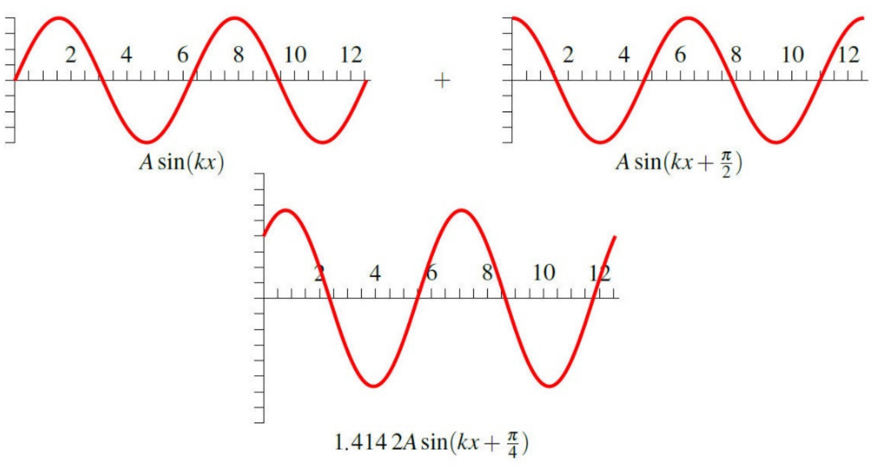


A wave with an amplitude twice as large as the original waves is obtained. This corresponds to constructive interference.

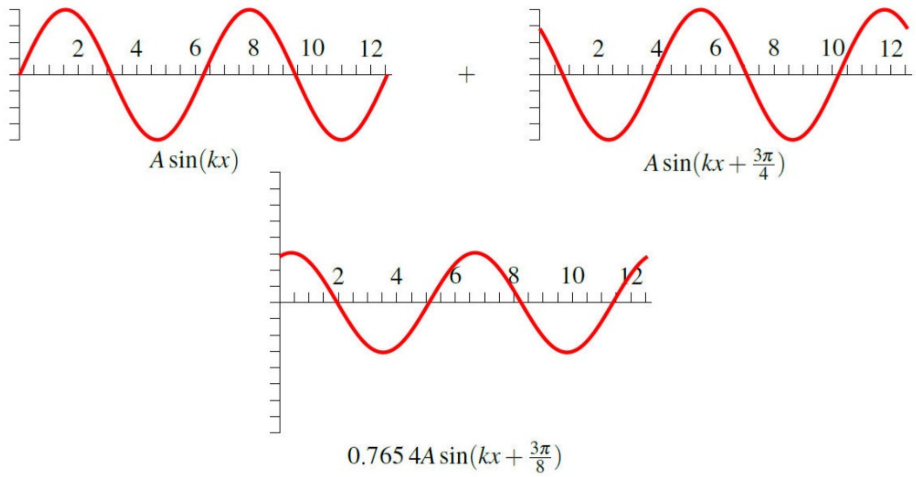
For two waves out of phase by one-eighth of a cycle ($\Delta\phi = \pi/4$), the result is



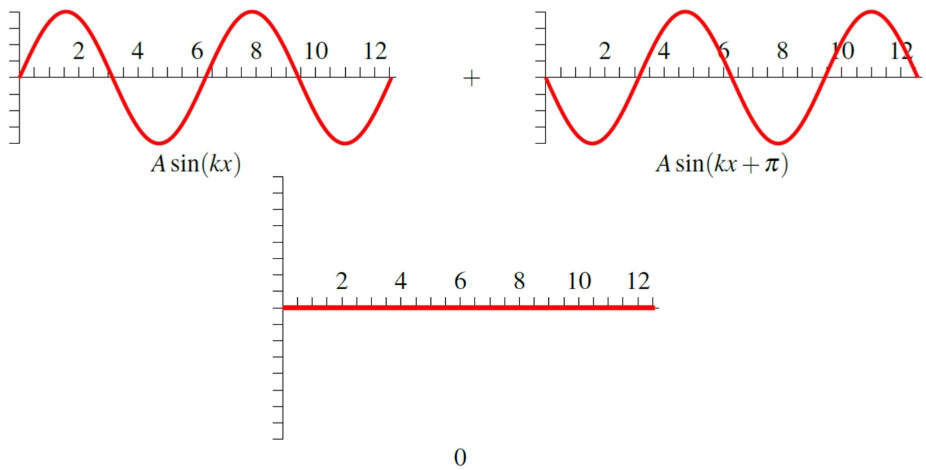
For two waves out of phase by one quarter of a cycle ($\Delta\phi = \pi/2$), the result is



For two waves out of phase by three-eighths of a cycle ($\Delta\phi = 3\pi/4$), the result is



For two waves out of phase by one half of a cycle ($\Delta\phi = \pi$), the result is



Thus, the amplitude decreased gradually until the amplitude is zero for a phase difference of π. In this case, the two waves cancel each other, and the medium does not oscillate at all. This corresponds to destructive interference.

Note that the sign of the phase difference does not matter since it is inside a cosine function. Also note that 2π can be added or subtracted to the phase difference as many times as one wants and the resulting amplitude stays the same. A 3π phase difference gives the same resulting amplitude as a π phase difference.

Resulting Amplitude for Two Waves With Different Amplitude

With two waves having different amplitudes

$$y_1 = A_1 \sin\left(kx - \omega t + \phi_1\right)$$
$$y_2 = A_2 \sin\left(kx - \omega t + \phi_2\right)$$

The resulting amplitude when these two waves are added is

Resulting Amplitude of Two Interfering Waves With Different Amplitude A_1 and A_2

$$A_{tot} = \sqrt{A_1^2 + 2A_1A_2\cos\left(\Delta\phi\right) + A_2^2}$$

If you want, you can see the proof of this formula in this document.
http://physique.merici.ca/waves/proofAdiff.pdf

Example 5.2.1

What is the resulting amplitude of the addition of these two waves?

$$y_1 = 4cm \cdot \sin\left(5\tfrac{rad}{m} \cdot x - 4\tfrac{rad}{s} \cdot t + 1rad\right) \quad \text{and} \quad y_2 = 3cm \cdot \sin\left(5\tfrac{rad}{m} \cdot x - 4\tfrac{rad}{s} \cdot t + 3rad\right)$$

The phase difference is

$$\begin{aligned} \Delta\phi &= 3rad - 1rad \\ &= 2rad \end{aligned}$$

Therefore, the amplitude is

$$\begin{aligned} A_{tot} &= \sqrt{A_1^2 + 2A_1A_2\cos\left(\Delta\phi\right) + A_2^2} \\ &= \sqrt{\left(4cm\right)^2 + 2 \cdot 4cm \cdot 3cm \cdot \cos\left(2rad\right) + \left(3cm\right)^2} \\ &= 3.8746cm \end{aligned}$$

When the waves do not have the same amplitude, constructive interference is obtained when the resulting amplitude has the greatest possible value and destructive interference is obtained when the resulting amplitude has the smallest possible value.

Necessary Conditions to Obtain Constructive or Destructive Interference

Constructive Interference

When there is constructive interference, the resulting amplitude is the greatest possible. This happens when

$$A_{tot} = \sqrt{A_1^2 + 2A_1A_2\cos\left(\Delta\phi\right) + A_2^2}$$

has its greatest value. This happens when the cosine is 1.

$$\cos \Delta\phi = 1$$

There are several solutions to this equation. These solutions are

$$\Delta\phi = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \pm 8\pi, \pm 10\pi, \ldots$$

These solutions can be written as

Condition to Obtain Constructive Interference

$$\Delta\phi = 2m\pi$$

where m is an integer.

Then, the amplitude is

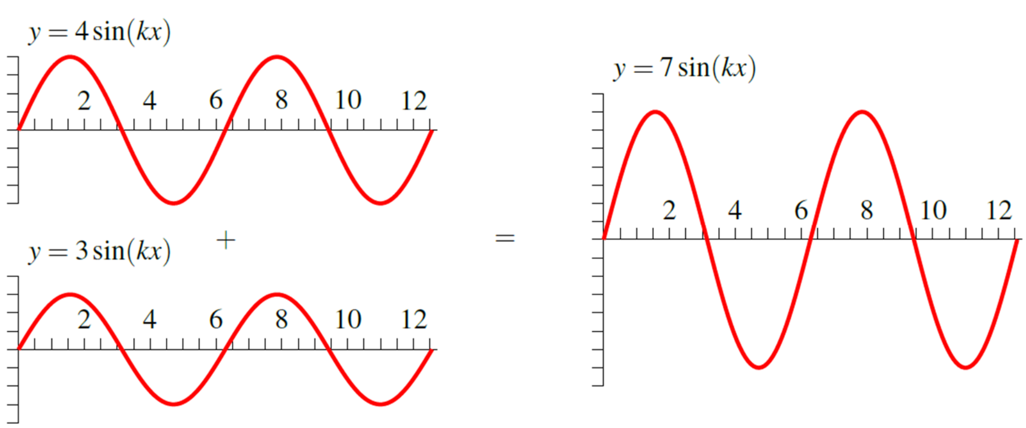
$$\begin{aligned} A_{tot} &= \sqrt{A_1^2 + 2A_1A_2 + A_2^2} \\ &= \sqrt{(A_1 + A_2)^2} \end{aligned}$$

This leads to

Amplitude When There Is Constructive Interference

$$A_{tot} = A_1 + A_2$$

For example, the resulting amplitude is 7 when these two waves make constructive interference.



__Destructive Interference__

When there is destructive interference, the resulting amplitude is the smallest possible. This happens when

$$A_{tot} = \sqrt{A_1^2 + 2A_1A_2\cos(\Delta\phi) + A_2^2}$$

has its smallest value. This happens when the cosine is -1.

$$\cos\Delta\phi = -1$$

There are several solutions to this equation. These solutions are

$$\Delta\phi = \pm\pi, \pm 3\pi, \pm 5\pi, \pm 7\pi, \pm 9\pi, \ldots$$

These solutions can be written as

Condition to Obtain Destructive Interference

$$\Delta\phi = (2m+1)\pi$$

where *m* is an integer

Then, the amplitude is

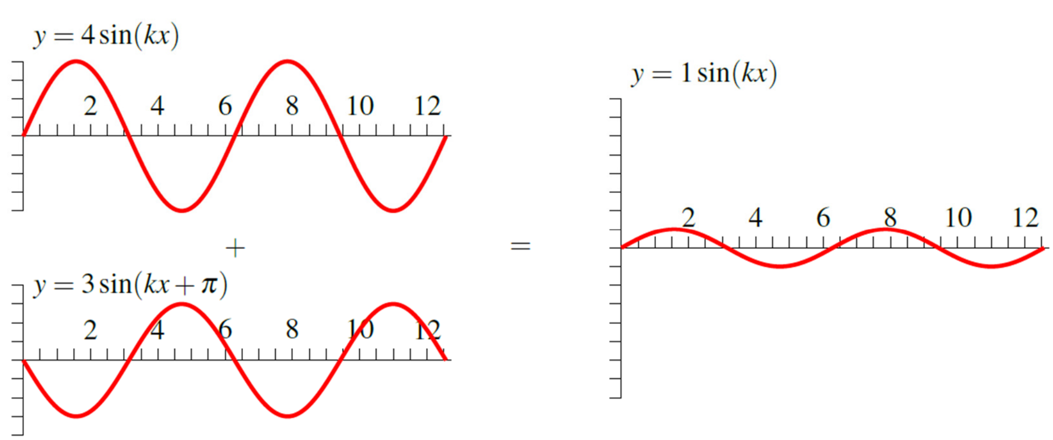
$$\begin{aligned} A_{tot} &= \sqrt{A_1^2 - 2A_1A_2 + A_2^2} \\ &= \sqrt{(A_1 - A_2)^2} \end{aligned}$$

Which leads to

Amplitude When There Is Destructive Interference

$$A_{tot} = |A_1 - A_2|$$

For example, in the following situation, a wave with an amplitude of 4 is making destructive interference with a wave with an amplitude of 3. The resulting wave has an amplitude of 1. This is the smallest amplitude that can be obtained by combining these two waves.



In this case, the amplitude does not vanish since the wave with the smallest amplitude cannot completely cancel the wave with the greatest amplitude.

5.3 PHASE DIFFERENCE BETWEEN TWO WAVES

The amplitude of the resulting wave depends on the phase difference between the interfering waves. But how can the phase difference between these two waves be known?

Three things can cause a phase difference between waves.

1) The waves do not take the same time to arrive at the observer ($\Delta\phi_T$).
2) The two sources emit waves out of phase from the beginning ($\Delta\phi_S$).
3) There are wave reflections ($\Delta\phi_R$).

The total phase difference is simply the sum of these 3 phase differences.

Phase Difference Between Two Waves

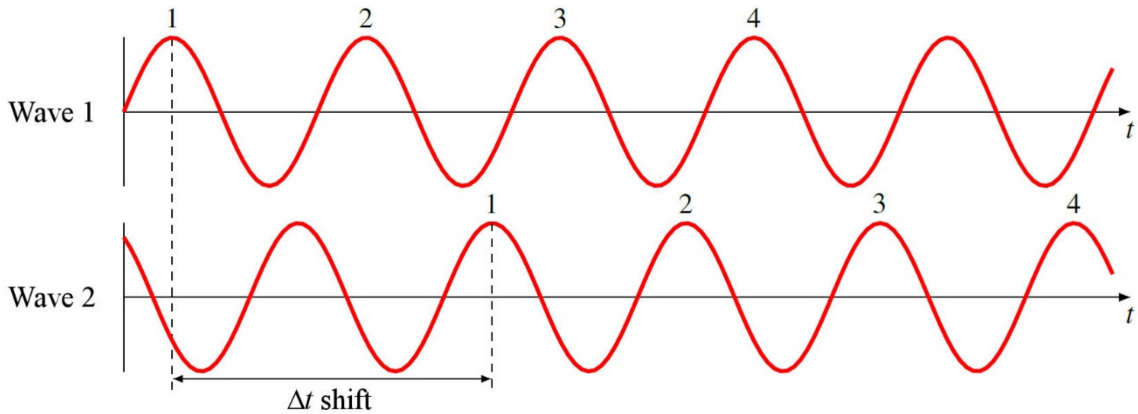
$$\Delta\phi = \Delta\phi_T + \Delta\phi_S + \Delta\phi_R$$

$\Delta\phi_T$ Calculation

Often, one of the waves takes more time to arrive at the observer than the other (like the wave coming from source 2 in the situation shown in the diagram). This difference in arrival time will be denoted Δt.

Let’s assume that the sources emit the waves in phase (i.e. they emit the wave maxima at the same time), but that the wave coming from source 2 takes a longer time to arrive at the observer than the wave coming from source 1.

The following graph shows the two oscillations received by the observer.



The sources emit maxima #1 at the same time, then maxima #2 at the same time, then maxima #3 at the same time, and so on. As it takes more time for the wave coming from source 2 to arrive at the observer, the maxima of the wave coming from source 2 will arrive Δt later compared to the maximum coming from source 1.

In this graph, the phase constant of wave 1 is 0 ($\phi_1 = 0$). Wave 2 is shifted towards the right by Δt compared to wave 1 since this wave arrives Δt later. Therefore, the phase constant of this wave is

$$\phi_2 = -\omega\Delta t$$

(Remember that phase constant is negative if the wave is shifted towards the positive values.) This means that the phase difference is

$$\begin{aligned}\Delta\phi_T &= \phi_2 - \phi_1 \\ &= -\omega\Delta t - 0 \\ &= -\omega\Delta t\end{aligned}$$

As $\omega = 2\pi/T$, the phase difference becomes

$\Delta\phi_T$ Calculation

$$\Delta\phi_T = -\omega\Delta t = -\frac{\Delta t}{T}2\pi$$

In this formula, Δt is the extra time it takes for the wave coming from source 2 to arrive at the observer. If the wave coming from source 2 arrives first, Δt is negative.

As the two waves travel at the same speed, the time it takes for each wave to reach the observer is

$$t_1 = \frac{r_1}{v} \qquad\qquad t_2 = \frac{r_2}{v}$$

where r_1 is the distance between source 1 and the observer and r_2 is the distance between source 2 and the observer. Then, the time difference is

$$\Delta t = \frac{r_2}{v} - \frac{r_1}{v}$$

$$\Delta t = \frac{r_2 - r_1}{v}$$

$$\Delta t = \frac{\Delta r}{v}$$

The phase difference thus becomes

$$\Delta\phi_T = -\frac{\Delta t}{T} 2\pi$$

$$\Delta\phi_T = -\frac{\Delta r}{vT} 2\pi$$

Since $vT = \lambda$, this gives

$\Delta\phi_T$ Calculation

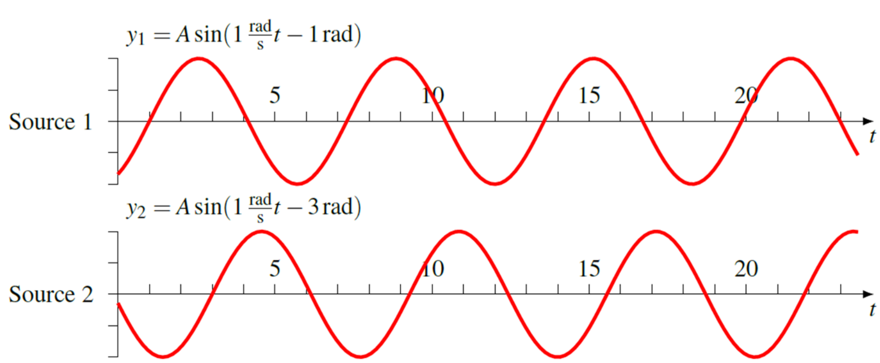
$$\Delta\phi_T = -\frac{\Delta r}{\lambda} 2\pi$$

In this formula, Δr is the extra distance the wave coming from source 2 has to travel to reach the observer ($\Delta r = r_2 - r_1$). If source 2 is closer than source 1, Δr is negative.

Δr is called *the path-length difference*.

$\Delta\phi_S$ Calculation

Sometimes, part of the phase difference comes from the fact that the sources do not emit their signals in phase. The following example illustrates this. Suppose that the two following graphs represent the wave emitted by two sources as a function of time.



The two phase constants can easily be identified to calculate the phase difference.

$$\begin{aligned}\Delta\phi_S &= \phi_{source\ 2} - \phi_{source\ 1}\\ &= -3rad - -1rad\\ &= -2rad\end{aligned}$$

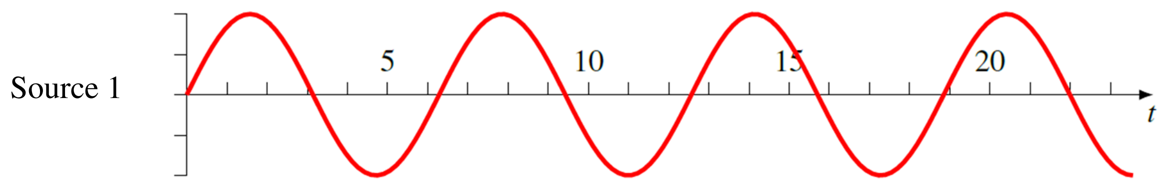
Therefore, the phase difference is

$\Delta\phi_S$ Calculation

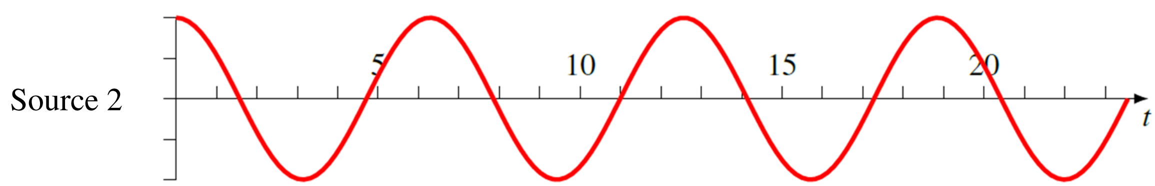
$$\Delta\phi_S = \phi_{source\ 2} - \phi_{source\ 1}$$

If source 2 is ahead of source 1, the value of $\Delta\phi_S$ is positive, and if source 2 lags behind source 1, the value of $\Delta\phi_S$ is negative (as in the example).

Sometimes the phase constants are not given, but it is said that wave 2 is shifted by a certain proportion of a cycle compared to the source 1. Again, let's take an example to illustrate this. Suppose that it is said that source 2 lags behind source 1 by ¾ of a cycle. In this case, any value can be used for the phase constant of source 1. Of course, a vanishing value makes things easier. The wave emitted by source 1 is then



Since source 2 lags behind this signal by ¾ of a cycle, its graphic should be



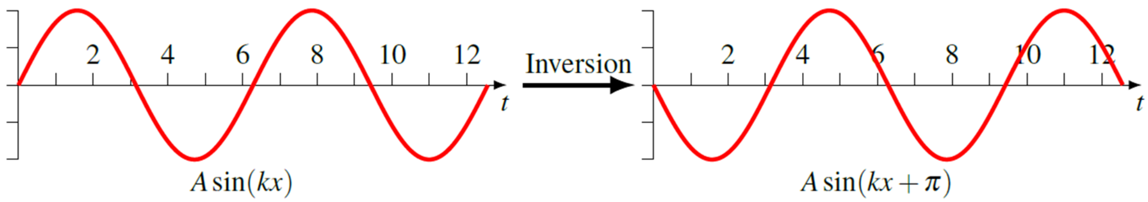
A shift of ¾ of a cycle towards the positive axis means that the phase constant is -3π/2.

This means that the phase difference

$$\begin{aligned}\Delta\phi_S &= \phi_2 - \phi_1\\ &= -\tfrac{3\pi}{2} rad - 0rad\\ &= -\tfrac{3\pi}{2} rad\end{aligned}$$

$\Delta\phi_R$ Calculation

When a wave is reflected, it can be inverted. This means that there could be a phase difference due to reflections since the inversion of a wave is equivalent to adding π to the phase constant of the wave, as seen in this diagram.



Thus, if the wave is inverted, the phase constant changes by π. This change of phase is denoted ϕ_R and it can take only the value 0 (not inverted) or π (inverted).

At the end, the difference between the phase changes of each wave must be made to obtain the phase difference. If the phase of each wave changes by π, the phase difference between them does not change.

Therefore, the phase difference due to reflection is

$\Delta\phi_R$ Calculation

$$\phi_R = 0 \text{ (not inverted) or } \pi \text{ (inverted)}$$

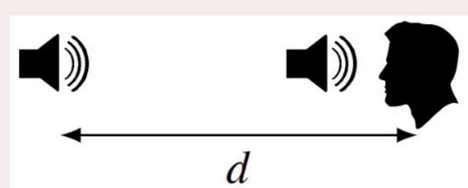
$$\Delta\phi_R = \phi_{R2} - \phi_{R1}$$

Sign of $\Delta\phi$

Note that the sign of total phase difference $\Delta\phi$ can always be reversed. The choice of which source is source 1 and which source is source 2 is entirely arbitrary. By inverting this choice, the sign of $\Delta\phi$ in inverted but this should logically have no impact on the results that will be obtained from the phase difference (such as the resulting amplitude of the wave).

Only the sign of the total phase difference $\Delta\phi$ can be changed. For example, the sign of $\Delta\phi_T$ cannot be change without also changing the sign of $\Delta\phi_S$ and $\Delta\phi_R$.

Example 5.3.1



These two speakers both emit sounds with a frequency of 500 Hz and amplitude A at the position where the observer is. The signal from the farthest speaker is ahead by ¼ cycle compared to the signal emitted by the other speaker. What is the amplitude of the wave received by the observer if the distance between the speakers is 3 m? (The speed of sound is 340 m/s.)

The resulting amplitude is given by

$$A_{tot} = \left| 2A\cos\left(\frac{\Delta\phi}{2}\right) \right|$$

Find this amplitude, the phase shift between the two waves must be known. The phase shift is

$$\Delta\phi = \Delta\phi_T + \Delta\phi_S + \cancel{\Delta\phi_R}$$

Here, the phase difference due to reflections vanishes since there is no reflection. The phase difference due to the time difference is

$$\Delta\phi_T = -\frac{\Delta r}{\lambda} 2\pi$$

As the wavelength is

$$\begin{aligned}\lambda &= \frac{340 \frac{m}{s}}{500Hz} \\ &= 0.68m\end{aligned}$$

the phase difference due to the time difference is

$$\begin{aligned}\Delta\phi_T &= -\frac{\Delta r}{\lambda} 2\pi \\ &= -\frac{-3m}{0.68m} \cdot 2\pi \\ &= 27.720rad\end{aligned}$$

By using -3 m for the difference in distance, source 2 was set as the speaker closest to the observer and source 1 as the speaker farthest to the observer. (Remember that Δr is negative if source 2 is closer.)

Now find the phase difference due to the sources. We'll set the phase constant of source 1 to zero. Then source 2 is lagging source 1 by ¼ of a cycle. This means that the sine of this wave of shifted ¼ of a cycle towards the right. This means that the phase constant of this source is -π/2. Thus, the phase difference due to the sources is

$$\begin{aligned}\Delta\phi_S &= \phi_{\text{source 2}} - \phi_{\text{source 1}} \\ &= -\frac{\pi}{2} - 0\end{aligned}$$

$$= -\frac{\pi}{2}$$

Therefore, the total phase difference is

$$\begin{aligned}\Delta\phi &= \Delta\phi_T + \Delta\phi_S \\ &= 27.720rad + -\frac{\pi}{2}rad \\ &= 26.149rad\end{aligned}$$

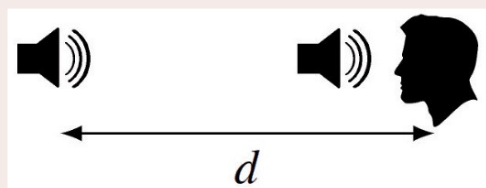
Thus, the amplitude is

$$\begin{aligned}A_{tot} &= \left|2A\cos\left(\frac{26.149rad}{2}\right)\right| \\ &= 1.747A\end{aligned}$$

It can be seen that we have a larger amplitude than with a single speaker, but the interference is not totally constructive (the amplitude would be then 2*A*).

Example 5.3.2

These two speakers both emit, in phase, sounds with a frequency of 500 Hz. What is the minimum distance between the speakers which allows for destructive interference at the position where the observer is? (The speed of sound is 340 m/s.)



In order to have destructive interference, the phase shift between the two waves must be

$$\Delta\phi = (2m+1)\pi$$

This phase difference is

$$\Delta\phi = \Delta\phi_T + \cancel{\Delta\phi_S} + \cancel{\Delta\phi_R}$$

Since there are no reflection and the phase difference between the sources is 0 (it is said that they are in phase), then the only phase difference is caused by the difference in time. As this phase difference is

$$\Delta\phi_T = -\frac{\Delta r}{\lambda}2\pi$$

the constructive interference condition becomes

$$-\frac{\Delta r}{\lambda}2\pi = (2m+1)\pi$$

This the path-length difference is

$$\Delta r = -(2m+1)\frac{\lambda}{2}$$

As the wavelength is 0.68 m (calculated in the previous example), the path-length difference is

$$\Delta r = -(2m+1)\cdot 0.34\,\mathrm{m}$$

m is an integer (be careful not to confuse it with the unit m (meters)). Using some values for m, the results are

if $m = 2$ Δr = -1.70 m
if $m = 1$ Δr = -1.02 m
if $m = 0$ Δr = -0.34 m
if $m = -1$ Δr = 0.34 m
if $m = -2$ Δr = 1.02 m

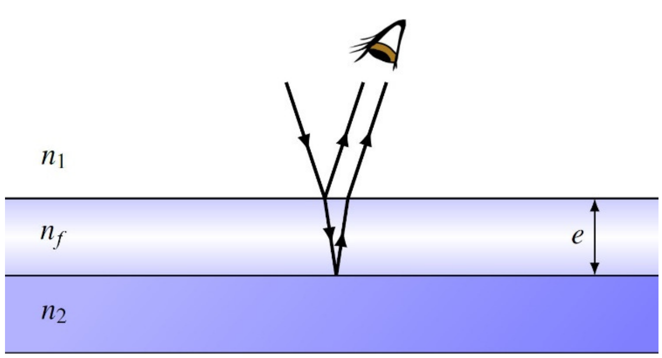
Thus, the smallest value for the distance between the speakers is 0.34 m (in fact, we were looking for the smallest distance in absolute value because the negative sign depends on our choice of which source is source 1. Here, there are two responses equal to 34 cm. Since the two signs are good here, source 2 can be put 34 cm in front or behind source 1).

5.4 THIN FILMS

Interference of the Reflections

Here is an example of the superposition of two waves having the same frequency and travelling in the same direction.

Light is shining on a surface on which there is a thin layer (also called a film) made of another substance. Part of the light is then reflected on one surface of the layer and another part on the other surface of the layer. These two reflections will then overlap and interfere. (Even if the light comes at a certain angle on the surface in the diagram, the only case considered here will be the case of light arriving perpendicularly to the surface of the thin layer.)



To determine the result of the interference, the phase difference between the two waves must be calculated. This phase difference is given by

$$\Delta\phi = \Delta\phi_T + \Delta\phi_S + \Delta\phi_R$$

Here, $\Delta\phi_S$ is zero because the two waves come from the same wave. There may be a $\Delta\phi_R$ since the two waves are reflected. The value of this phase difference will depend on the refractive indices. $\Delta\phi_T$ is fairly easy to determine because the time difference is the time it takes for the wave reflected on the second surface to pass through the film twice. This time is

$$\Delta t = \frac{\text{distance}}{\text{speed}} = \frac{2e}{v}$$

where e is the thickness of the film. Do not forget that in a substance having a refractive index n_f (where f stands for film, a synonym of thin layer) the speed of light is

$$v = \frac{c}{n_f}$$

The difference in arrival time is thus

$$\Delta t = \frac{2e}{v}$$
$$\Delta t = \frac{2n_f e}{c}$$

If wave 1 is chosen as being the one making the round trip in the film, Δt is negative. Thus, the phase difference is

$$\begin{aligned}\Delta\phi_T &= -\frac{\Delta t}{T}2\pi \\ &= -\frac{-2n_f e}{cT}2\pi\end{aligned}$$

Since cT is equal to the wavelength of the light in vacuum (or in air, it is not that different), that is denoted λ_0, the phase difference becomes

$$\Delta\phi_T = \frac{4\pi n_f e}{\lambda_0}$$

The total phase difference is thus

$$\Delta\phi = \Delta\phi_T + \Delta\phi_R$$

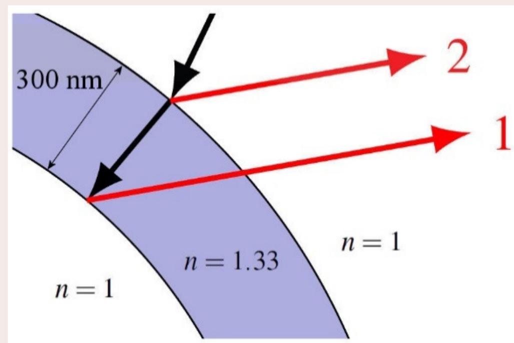
Using the value of $\Delta\phi_T$, this equation is

Phase Difference Between the Two Waves Reflected in Thin Films

$$\Delta\phi = \frac{4\pi n_f e}{\lambda_0} + \Delta\phi_R$$

where $\Delta\phi_R$ can only be 0, $-\pi$, or π depending on the refractive indices.

Example 5.4.1



A soap bubble floats in air. The index of refraction of the wall of the bubble is 1.33, and the thickness of the wall is 300 nm.

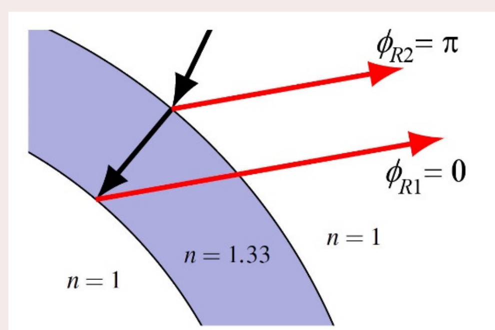
a) What are the wavelengths of the visible light that are strongly reflected by the bubble?

The wavelengths that are strongly reflected are the ones for which there is constructive interference. This means that the phase difference between the two waves must be

$$\Delta\phi = 2m\pi$$

In addition, the total phase shift in this situation is

$$\Delta\phi = \frac{4\pi n_f e}{\lambda_0} + \Delta\phi_R$$



$\Delta\phi_R$ must then be found. As wave 1 (the one with the longest path) is travelling in water and is reflected on a medium having a smaller index (air), the wave is not inverted and undergoes no phase shift ($\phi_{R1} = 0$). As wave 2 (the one with the shortest path) is travelling in air and is reflected on a medium having a larger index of refraction (water), the wave is inverted and undergoes a phase shift equal to π ($\phi_{R2} = \pi$). The difference between these two phase shifts is

$$\begin{aligned}\Delta\phi_R &= \phi_{R2} - \phi_{R1}\\ &= \pi - 0\\ &= \pi\end{aligned}$$

The total phase difference is, therefore,

$$\Delta\phi = \frac{4\pi n_f e}{\lambda_0} + \pi$$

With the condition for constructive interference, we obtain

$$2m\pi = \frac{4\pi n_f e}{\lambda_0} + \pi$$

If this equation is solved for λ_0, it becomes

$$\begin{aligned}\lambda_0 &= \frac{4en_f}{2m-1}\\ &= \frac{4\cdot 300nm\cdot 1.33}{2m-1}\\ &= \frac{1596nm}{2m-1}\end{aligned}$$

Using different integer values for *m*, the wavelengths are

λ_0 = 1596 nm for $m = 1$
λ_0 = 532 nm for $m = 2$
λ_0 = 319.2 nm for $m = 3$

Other values of *m* give smaller wavelengths. The only wavelength in the visible spectrum making constructive interference is, therefore, 532 nm. This wavelength will be very bright in the reflected light.

b) What are the wavelengths of the visible light that are weakly reflected by the bubble?

The wavelengths absent in the reflected light are those for which there is destructive interference. The condition for destructive interference is

$$\Delta\phi = (2m+1)\pi$$

Using the formula for the phase difference obtained previously, we have

$$(2m+1)\pi = \frac{4\pi n_f e}{\lambda_0} + \pi$$

If this equation is solved for λ_0, it becomes

$$\begin{aligned}\lambda_0 &= \frac{4en_f}{2m}\\ &= \frac{1596nm}{2m}\end{aligned}$$

Using different integer values for m, the wavelengths are

$$\lambda_0 = 798 \text{ nm for } m = 1$$
$$\lambda_0 = 399 \text{ nm for } m = 2$$
$$\lambda_0 = 266 \text{ nm for } m = 3$$

Other values of m give smaller wavelengths. The only wavelength in the visible spectrum for which there is destructive interference is, therefore, 399 nm. (Actually, it is a bit outside of the visible spectrum, but just barely.) This wavelength will have a weak intensity in the reflected light.

Here is a beautiful picture of light interference in soap bubble walls.

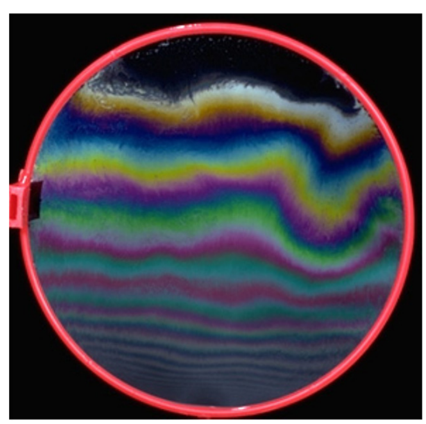
www.pinterest.com/pin/521080619356053532/

There are different colours at different places since the equation for the phase difference in the bubble wall

$$\Delta\phi = \frac{4\pi n_p e}{\lambda_0} + \pi$$

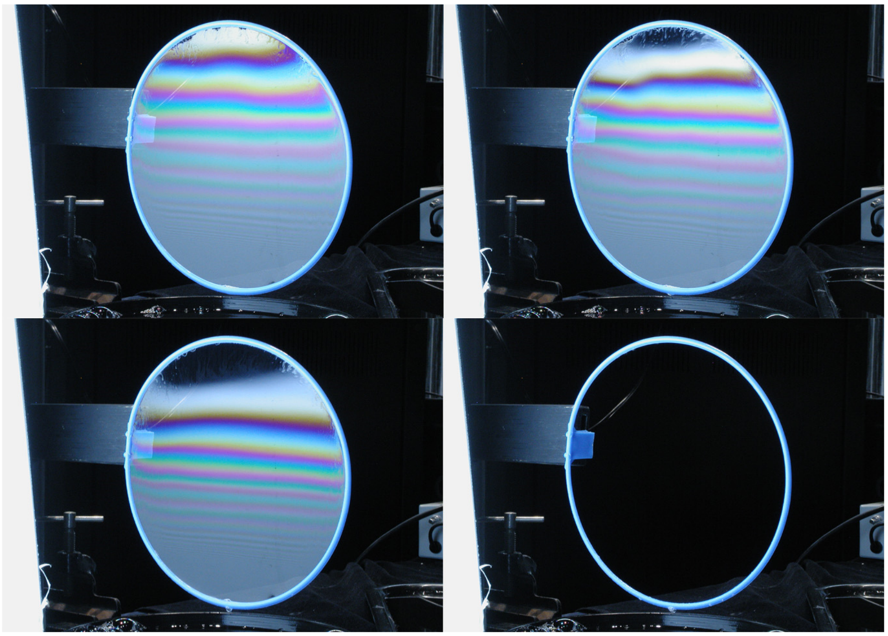
indicates that the total phase difference depends on the thickness of the film and on the wavelength. Thus, depending on the thickness of the film, some colours make constructive interference and are more intense than what would be seen without the thin film. Other colours make destructive interference and are absent from the light reflected by the film. (The result also depends on the angle of incidence of the light on the layer. Here, calculations were made only for light having a 90° incidence angle. A more general treatment would have shown that the result of the interference also depends on the angle of incidence of the light on the film.)

On the right, an image of the reflection made by a thin film can be seen. This is a thin film made of soap which is thicker at the bottom than at the top. The result of the interference is not the same depending on the thickness of the film. By varying the thickness, the colours that make constructive interference changes and the colour of the reflected light then changes.



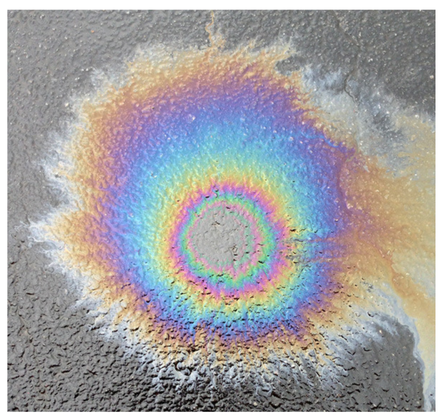
www.compadre.org/informal/index.cfm?Issue=52

The phase difference equation also indicates that if the bubble becomes very thin (meaning that the thickness is much smaller than the wavelength), the phase shift is simply π. Such a phase difference means that there is destructive interference for every colour and that there is no reflected light. That is what can be seen in the next image.



isites.harvard.edu/fs/docs/icb.topic186203.files/images/ThinFilmInterference02.jpg

As the soap flows down, the soap film becomes thinner and thinner. As this happens, the film becomes so thin that there is destructive interference for every colour and there is no reflected light anymore. This phenomenon appears at the top of the wall on the second image and expands towards the bottom in the following images. On the last image, the entire wall is very thin and there is no reflected light anywhere. Here's a clip of this experiment.
http://www.youtube.com/watch?v=lRhUQTuEu3I



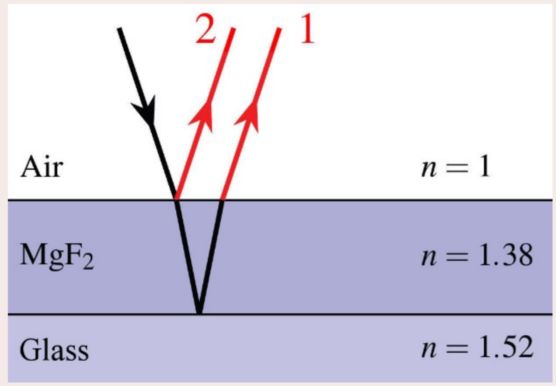
Light interference in a thin film is also responsible for the different colours of oil floating on water. Oil does not have this colour, it seems to have these colours because some colours undergo constructive interference in the thin film formed by the oil on the water and are, therefore, strongly amplified in the reflected light.

hyperphysics.phy-astr.gsu.edu/hbase/phyopt/oilfilm.html

Unwanted reflections can also be prevented with a thin film, as the following example shows.

Example 5.4.2

A thin film of magnesium fluoride (n = 1.38) is used to coat a glass surface (n = 1.52) to form an anti-reflective film. What minimum thickness should the magnesium fluoride film have to suppress light reflection?



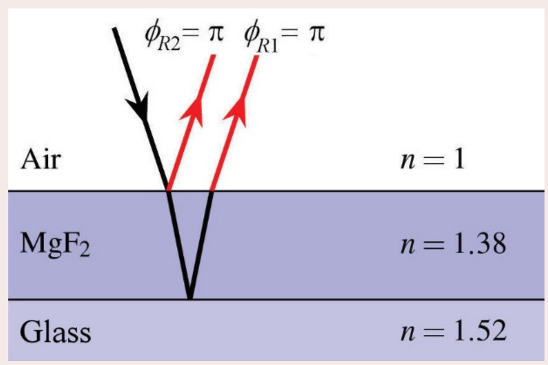
With an anti-reflection film, reflected light having a low intensity is sought. This means that there must be destructive interference in the reflected light. Thus, the phase shift between the two waves must be

$$\Delta\phi = (2m+1)\pi$$

In addition, the total phase shift for a thin film is

$$\Delta\phi = \frac{4\pi n_f e}{\lambda_0} + \Delta\phi_R$$

$\Delta\phi_R$ must then be found. As wave 1 (the one with the longest path) is travelling in magnesium fluoride and is reflected on a medium having a larger index (glass), the wave is inverted and undergoes a phase shift ($\phi_{R1} = \pi$). As wave 2 (the one with the shortest path) is travelling in air and is reflected on a medium having a larger index of refraction (magnesium fluoride), the wave is inverted and undergoes a phase shift equal to π ($\phi_{R2} = \pi$). The difference between these two phase shifts is



$$\begin{aligned}\Delta\phi_R &= \phi_{R2} - \phi_{R1} \\ &= \pi - \pi \\ &= 0\end{aligned}$$

Therefore, the total phase shift is

$$\Delta\phi = \frac{4\pi n_f e}{\lambda_0}$$

Using the condition for destructive interference, the following result is obtained

$$(2m+1)\pi = \frac{4\pi n_f e}{\lambda_0}$$

Solving for the thickness e, the result is

$$e = \frac{(2m+1)\lambda_0}{4n_f}$$

Unfortunately, the thickness depends on the wavelength, and it is, therefore, impossible to suppress reflections for every colour in the visible spectrum. So, a 550 nm wavelength, which is in the middle of the visible spectrum, will be used.

The minimum value of the thickness (found with $m = 0$) is

$$\begin{aligned} e_{\min} &= \frac{\lambda_0}{4n_f} \\ &= \frac{550nm}{4 \cdot 1.38} \\ &= 99.6nm \end{aligned}$$

(Note that with this thickness, the wavelengths for which there is constructive interference are given by

$$\begin{aligned} \lambda_0 &= \frac{2en_f}{m} \\ &= \frac{275nm}{m} \end{aligned}$$

None of these wavelengths are in the visible spectrum, which is good for our anti-reflective film. Destructive interference is not perfect for the other colours that are not at 550 nm. The further the wavelength is from 550 nm, the greater the intensity of the reflection is. At each end of the spectrum, red light (700 nm) has an intensity of 0.43 I and violet light (400 nm) has an intensity of 1.23 I (if it is assumed that the amplitudes of reflected waves are identical, which is unlikely). The reflection of violet light is thus more intense than the intensity there would be without a film (I). This is why a piece of glass coated with such an anti-reflective thin film has a light-violet hue.)

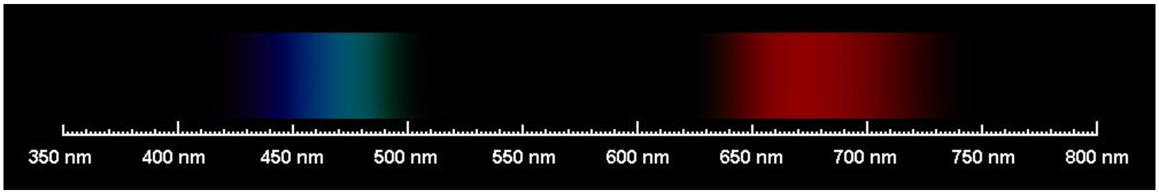
Why Must the Film Be Thin?

Let's use the result of the previous example and calculate the wavelengths for which there is constructive interference for two different film thicknesses. For a 500 nm thickness, the wavelengths for which there is constructive interference are given by

$$\lambda_0 = \frac{2en_f}{m} = \frac{1380nm}{m}$$

With this formula, the following wavelengths are obtained: 1380 nm ($m = 1$), 690 nm ($m = 2$), 460 nm ($m = 3$), 345 nm ($m = 4$),…

There are only 2 wavelengths in the visible spectrum, and they are well separated. The spectrum of the reflected light would look like this.



www.euhou.net/index.php/exercises-mainmenu-13/classroom-experiments-and-activities-mainmenu-186/179-observations-of-various-spectra-with-a-home-made-spectroscope

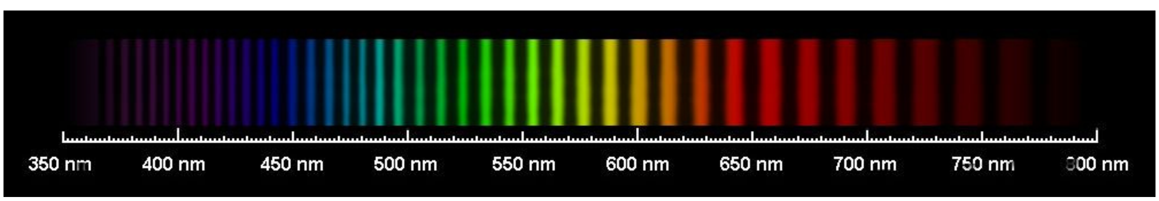
It is easy to see that this spectrum corresponds to a light that is different from white light, which has a continuous spectrum.

With a 10,000 nm thickness (actually, that's not that thick as this is only 0.01 mm), the wavelengths for which there is constructive interference are given by

$$\lambda_0 = \frac{2en_f}{m} = \frac{27600nm}{m}$$

With this formula, the following wavelengths are obtained: 27 600 nm (m = 1), 13 800 nm (m =2),…, 552 nm (m = 50), 541 nm (m = 51), 530 nm (m = 52),…

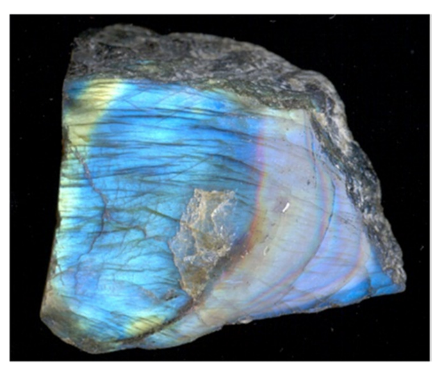
Those are only a few of the 30 wavelengths in the visible spectrum for which there is constructive interference, separated by only 10 nm on average. The spectrum of the reflected light then looks like this.



It this case, it is very difficult to see that there is interference because the maxima are too close to each other. There are so many colours in the visible spectrum for which there is constructive interference that it is hard to distinguish this spectrum from a continuous spectrum (which is what would be observed if there were no interference).

Thin Films in Nature

Thin films (or something similar) can be found in several places in nature. Labradorite is a mineral in which interference occurs because there are thin layers in the rock. The interference amplifies the reflection of blue light and gives this peculiar look to the rock.



geology.about.com/od/gems/ig/gemeffects/labradorescence.htm

A phenomenon similar to thin film interference also occurs to create a blue hue in butterfly wings and peacock feathers.

imagesbackgrounds.in/Peacock-001-peacock.html and bleuocean.centerblog.net/2220-papillon-bleu

Blue and green light reflections are amplified by interference in some places to produce brighter colours.

The following clip shows how interference is produced in butterflies' wings.
http://www.youtube.com/watch?v=3lbrjJpO9b4

The following clip shows how the properties of the thin film in a butterfly wing can be changed.
http://www.youtube.com/watch?v=jeUd_ittNns

5.5 SUPERPOSITION OF TWO WAVES WITH DIFFERENT FREQUENCIES

Now two waves going in the same direction but having different frequencies will be added. In fact, the oscillation at a specific place of the medium in which there are two waves are travelling will be considered.

Result of the Superposition

The waves, having frequencies f_1 and f_2, both generates oscillations at the position where the observer is. For simplicity, it will be assumed that the waves are in phase at t = 0 so that the oscillations are given by

$$y_1 = A\sin(\omega_1 t)$$
$$. \; y_2 = A\sin(\omega_2 t) .$$

The result of the passage of these two waves is obtained by adding the displacement made by each wave. This sum is

$$\begin{aligned} y_{tot} &= y_1 + y_2 \\ &= A\sin(\omega_1 t) + A\sin(\omega_2 t) \\ &= A\left(\sin(\omega_1 t) + \sin(\omega_2 t)\right) \end{aligned}$$

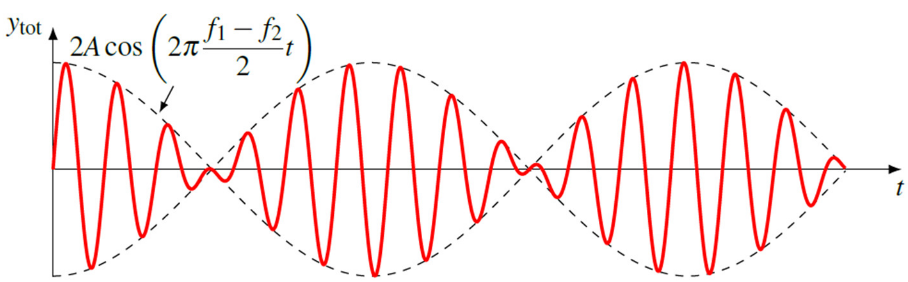
To simplify, the following identity is used

$$\sin\theta_1 + \sin\theta_2 = 2\cos\frac{\theta_1 - \theta_2}{2}\sin\frac{\theta_1 + \theta_2}{2}$$

The total displacement then becomes

$$\begin{aligned} y_{tot} &= 2A\cos\frac{\omega_1 t - \omega_2 t}{2}\sin\frac{\omega_1 t + \omega_2 t}{2} \\ &= 2A\cos\left(2\pi\frac{f_1 - f_2}{2}t\right)\sin\left(2\pi\frac{f_1 + f_2}{2}t\right) \end{aligned}$$

The graph of this function makes it even more obvious.



The result is an oscillation (solid line) with an amplitude that changes with time (dotted line). The equation can, therefore, be interpreted as follows.

$$y = \underbrace{2A\cos\left(2\pi\frac{f_1 - f_2}{2}t\right)}_{\text{changing amplitude}}\underbrace{\sin\left(2\pi\frac{f_1 + f_2}{2}t\right)}_{\text{oscillation}}$$

According to the equation, the frequency of the oscillation is

Frequency of the Oscillation With Two Waves of Different Frequencies

$$f_{osc} = \frac{f_1 + f_2}{2}$$

The frequency of the amplitude variations is found with the frequency of the cosine function in the amplitude. However, this frequency must be doubled because there are two maxima of the sound for each cycle since the sound is maximum when the cosine is 1 and when the cosine is -1. Remember that a negative sign in the amplitude is a hidden phase constant and is not part of the amplitude. Therefore, an amplitude of -2*A* is really an amplitude of 2*A*. The frequency is thus

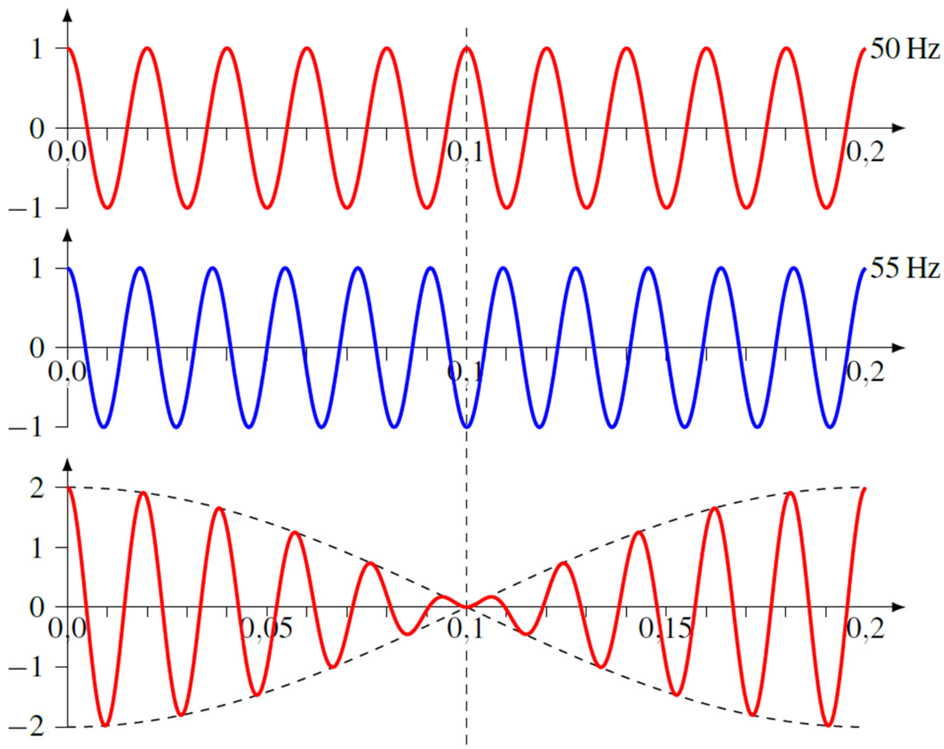
Frequency of the Changes of Amplitude

$$f_{ampli} = \left|f_1 - f_2\right|$$

There is an absolute value because a negative frequency has no meaning. In the formula, this absolute value was automatically done since $\cos\theta = \cos -\theta$.

This phenomenon can be understood with the following example: a 50 Hz wave and a 55 Hz wave are superimposed. Suppose that the crests of each wave are received initially. At this time, the two waves add up to generate constructive interference (large amplitude on the left of the bottom graph).

(Beware, this graph does not show the shape of the wave, it shows the oscillation of a molecule of the medium as a function of the time.)



0.1 seconds later, exactly 5 cycles have been received in 0.1 seconds for the 50 Hz wave. Thus, if a crest was received at $t = 0$ s, a crest of the wave is also received at $t = 0.1$ s since this is exactly 5 cycles later. On the other hand, 5.5 cycles have been received in 0.1 seconds for the 55 Hz wave. If a crest was being received at $t = 0$ s, a trough is being received at $t = 0.1$ s (5 cycles + a half cycle). The crest of the 50 Hz wave is now cancelled by the trough of the 55 Hz wave and there is destructive interference (vanishing amplitude in the middle of the bottom graph).

0.1 seconds later, the amplitude is great again. A crest is received at $t = 0.2$ s for the 50 Hz wave since a crest was received at $t = 0.1$ s and this is exactly 5 cycles later. On the other hand, 5.5 cycles have been received in 0.1 seconds for the 55 Hz wave. If a trough was being received at $t = 0.1$ s, then a crest must be received at $t = 0.2$ s (5 cycles + a half cycle). The crest of the 50 Hz wave is now added to the crest of the 55 Hz wave and there is constructive interference again. This process is constantly repeating, always alternating between constructive and destructive interference.

Beats

When two sounds with frequencies close to each other (less than 20 Hz difference) are played simultaneously, a sound with a varying intensity is heard. The sound disappears and reappears constantly. If the frequencies are nearer to one another, the intensity of the sound changes more slowly. These changes of intensity are called ***beats***.
http://www.youtube.com/watch?v=5hxQDAmdNWE

Listen to this demonstration. A sound with a frequency of 1000 Hz is played first, then a 1004 Hz sound is played, and, finally, the two sounds are played at the same time. Beats are then heard.
http://physique.merici.ca/ondes/sons/beats.mp3

These changes of intensity are exactly the ones predicted by the equations. The sound heard has the frequency of the oscillations and the beat frequency corresponds to the frequency of the change of amplitude.

Beats

$$f_{sound} = \frac{f_1 + f_2}{2} \qquad\qquad f_{beats} = \left| f_1 - f_2 \right|$$

The frequency difference should be less than 20 Hz (approximately). If the frequency difference is greater than 20 Hz, the changes are too fast and they cannot be perceived. The average intensity will be perceived if the beat frequency is too high.

Example 5.5.1

A 440 Hz sound and a 444 Hz sound are played at the same time.

a) What is the frequency of the sound heard?

$$\begin{aligned} f_{sound} &= \frac{440Hz + 444Hz}{2} \\ &= 442Hz \end{aligned}$$

b) What is the frequency of the beats?

$$\begin{aligned} f_{beats} &= 444Hz - 440Hz \\ &= 4Hz \end{aligned}$$

Therefore, a 442 Hz sound is heard, and its intensity reaches its maximum 4 times per second.

Radar Gun

The superposition of waves (microwaves) of different frequencies is used by policemen to measure the speed of cars with a radar gun.

The radar emits microwaves which are reflected on the car to come back to the radar. As the car move, the frequency of the waves receive by the radar gun is shifted due to the Doppler effect.

www.laserveil.com/police/radar/

As a result, the wave returning to the gun does not have the frequency as the wave emitted by the radar gun. The difference in frequency is measured by superimposing the initial wave and the received wave. The frequency of the change in the amplitude of the resulting wave gives the difference in frequency.

As the frequency change depends on the speed of your car, the device can calculate the speed of your vehicle from the frequency change.

Example 5.5.2

A policeman sends microwaves with a 20 GHz frequency towards a car. The frequency of amplitude variation obtained by superimposing the emitted and reflected waves received by the radar is 5,000 Hz. What is the speed of the car?

Since the amplitude varies with a frequency of 5000 Hz, then the difference in frequency is

$$\Delta f = 5000 Hz$$

Now let's calculate this frequency shift with the Doppler effect.

First, we will calculate the frequency of the waves received by the car. According to the Doppler effect formula, this frequency is

$$f' = f \frac{c + v_{car}}{c}$$

(Since the car is heading towards the radar gun, the speed of the car is negative in the Doppler Effect formula, which gives us the positive sign in the formula.)

After the reflection, the car becomes a moving source emitting the frequency f', and the radar becomes the observer (at rest). The frequency received by the radar is once again shifted by this second Doppler Effect. The frequency becomes

$$f'' = f' \frac{c}{c - v_{car}}$$

The frequency received is thus

$$f'' = f \frac{c + v_{car}}{c} \frac{c}{c - v_{car}} = f \frac{c + v_{car}}{c - v_{car}}$$

Thus

$$\begin{aligned}
\Delta f &= f \frac{c + v_{car}}{c - v_{car}} - f \\
&= f \frac{c - v_{car} + 2v_{car}}{c - v_{car}} - f \\
&= f \left(1 + \frac{2v_{car}}{c - v_{car}} \right) - f \\
&= \frac{2 f v_{car}}{c - v_{car}}
\end{aligned}$$

As the speed of the car is negligible compared to the speed of light, the shift becomes

$$\Delta f = \frac{2 f v_{car}}{c}$$

Since we know that the frequency difference is 5000 Hz, we have

$$\begin{aligned}
5000 Hz &= \frac{2 \cdot 20 \times 10^9 Hz \cdot v_{car}}{3 \times 10^8 \frac{m}{s}} \\
v_{car} &= 37.5 \tfrac{m}{s} \\
v_{car} &= 135 \tfrac{km}{h}
\end{aligned}$$

5.6 STANDING WAVES

Now, two waves having the same frequency and the same amplitude but moving in opposite directions will be added. Often, this happens when a wave is reflected. This leads to a superposition of a wave travelling in one direction before the reflection and a wave going in the other direction after the reflection.

Superposition of Two Identical Waves Travelling in Opposite Directions

The wave moving towards the positive x-axis is described by

$$y_1 = A\sin\left(kx - \omega t\right)$$

and the wave travelling towards the negative x-axis (same amplitude and wavelength) is described by

$$y_2 = A\sin\left(kx + \omega t\right)$$

Vanishing phase constants were used to simplify. The same results would be obtained if non-vanishing constant were used. The sum of the two waves is

$$y_{tot} = A\left[\sin\left(kx - \omega t\right) + \sin\left(kx + \omega t\right)\right]$$

To write this equation into another form, the following trigonometric identity is used

$$\sin\theta_1 + \sin\theta_2 = 2\sin\tfrac{1}{2}\left(\theta_1 + \theta_2\right)\cos\tfrac{1}{2}\left(\theta_1 - \theta_2\right)$$

The sum then becomes

$$y_{tot} = 2A\sin\tfrac{1}{2}\left(\left(kx - \omega t\right) + \left(kx + \omega t\right)\right)\cos\tfrac{1}{2}\left(\left(kx - \omega t\right) - \left(kx + \omega t\right)\right)$$

Simplifying, the following result is obtained.

Standing Wave Equation

$$y_{tot} = 2A\sin kx\cos\omega t$$

Here is the interpretation of this equation.

$$y_{tot} = \underbrace{2A\sin kx}_{\text{Amplitude that depends on the position}} \times \underbrace{\cos\omega t}_{\text{oscillation as a function of time}}$$

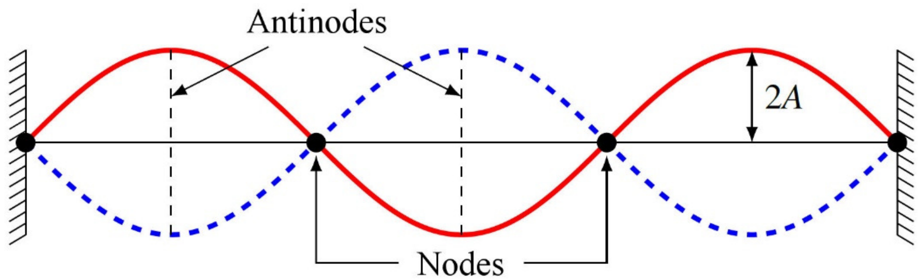
Each piece of the medium, therefore, makes an oscillation with an amplitude that depends on the position.

Oscillation Amplitude for a Standing Wave

$$A_{tot} = \left|2A\sin kx\right|$$

The amplitude is the absolute value since a negative value for the amplitude is simply a hidden π radians phase shift.

If these are waves on a string, these equations represent a rope that oscillates between the farthest position (curved black line) from its equilibrium position and at its other farthest position (curved grey line) from its equilibrium position.



On the rope, there are nodes (where the rope does not oscillate) and antinodes (where there is an oscillation).

Those are ***standing waves***. In this type of wave, the medium oscillates (except at the nodes), but the crests and nodes of the wave do not travel along the rope.

The amplitude of oscillation reaches its maximum value ($2A$) at the centre of the antinodes. The amplitude of oscillation reaches its minimum value (0) at the nodes. Remember that A is the amplitude of each of the waves forming the standing wave and not the amplitude of the standing wave, which is A_{tot}.

Here is a clip showing standing waves. A wave was sent in a medium and this wave is reflected when it arrives at the other end of the medium. There are then two superimposed waves travelling in opposite directions. In some places, the material oscillates with some amplitude and in other places, there is no oscillation at all (the amplitude is zero).
http://www.youtube.com/watch?v=jovIXzvFOXo
The following animation also shows a standing wave.
https://youtu.be/kmL7v1Cbl50

Standing Wave on a Rope

Consider a taut rope fixed to something at each end. Then, no motion whatsoever is possible at each end of the rope. This means that a node of the wave must be located at each end of the rope ($x = 0$ and $x = L$).

At $x = 0$, the amplitude of the wave is already zero because

$$A_{tot} = \left|2A\sin\left(k\cdot 0\right)\right| = 0$$

(This is nice but we arrive at this result because vanishing phase constants have been chosen previously. If non-vanishing phase constants had been used, a relation concerning these phase constants would have been obtained to have a node at $x = 0$.)

To have a node at $x = L$, the following equation must hold.

$$\begin{gathered}
\left|2A\sin kL\right| = 0 \\
\sin kL = 0 \\
kL = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, \ldots
\end{gathered}$$

The last line is obtained by finding all possible solutions of the inverse sine function. Negative solutions do not make sense since the length of the rope cannot be negative. Forgetting these negative solutions, this result can be written in the following way.

$$kL = n\pi \quad (n \text{ is a positive integer})$$

This means that

$$\begin{gathered}
kL = n\pi \\
\frac{2\pi}{\lambda} L = n\pi \\
\lambda = \frac{2L}{n}
\end{gathered}$$

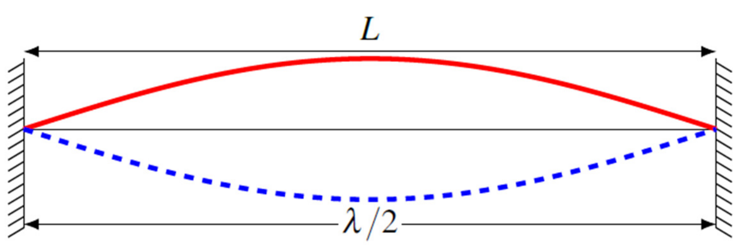
Therefore, the wavelength of a standing wave can have only certain values. Many values are still possible, though. To help identify these solutions, an index is added to the wavelength. Thus, λ_3 is the value of the wavelength for $n = 3$.

The formula for the possible wavelength of a standing wave is then

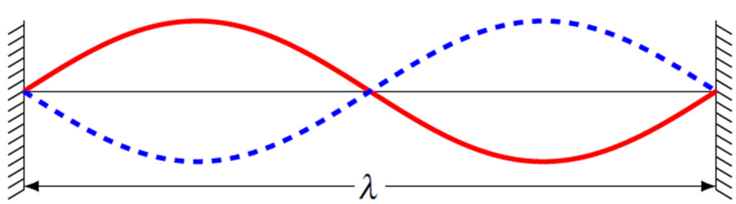
Possible Wavelengths of a Standing Wave on a Rope

$$\lambda_n = \frac{2L}{n} \quad (n \text{ is a positive integer})$$

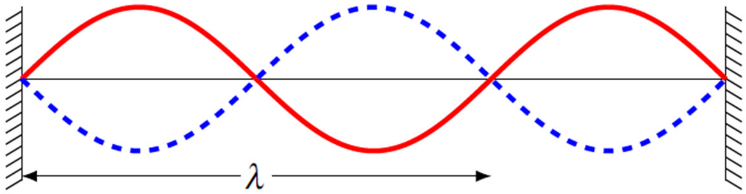
Here are some of these possibilities.



$n = 1$
1st mode = 1st harmonic
$\lambda = 2L$

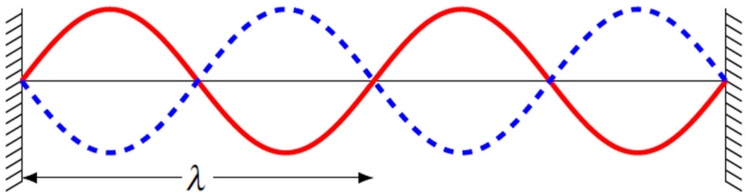


$n = 2$
2nd mode = 2nd harmonic
$\lambda = L$



$n = 3$
3rd mode = 3rd harmonic

$$\lambda = \frac{2L}{3}$$



$n = 4$
4th mode = 4th harmonic

$$\lambda = \frac{L}{2}$$

Those different solutions are called **harmonics** or **modes**. The standing wave with $n = 5$, for example, is called the 5th harmonic or the 5th mode.

Note that the distance between the nodes is always equal to half the wavelength.

Distance Between the Nodes of a Standing Wave

$$\text{distance} = \frac{\lambda}{2}$$

If only some values of the wavelength are possible, then only some values of the frequency are possible. The possible frequencies are found with

$$f = \frac{v}{\lambda}$$

Using the formula for the possible wavelength, the possible frequencies are given by

Possible Frequencies of a Standing Wave on a Rope

$$f_n = \frac{nv}{2L}$$

The frequency of the fundamental mode (f_1) is called the *fundamental frequency.* Since

$$f_1 = \frac{v}{2L}$$

the possible frequencies formula can also be written as

Possible Frequencies of a Standing Wave on a Rope

$$f_n = nf_1$$

It can be noted that the frequency increases for higher harmonics. Let's return to the video previously seen to observe the different harmonics of a standing wave. Notice how the frequency of oscillation of the hand increases to create higher-order modes.
http://www.youtube.com/watch?v=jovIXzvFOXo

If the person were to try to generate a wave with a frequency that does not correspond to a possible frequency of a standing wave, the medium would move, but this motion would not be a standing wave. A more chaotic motion would then be obtained.

For a standing wave on a string, the formula obtained for the speed can be used to get

Possible Frequencies of a Standing Wave on a Rope

$$f_n = \frac{n}{2L}\sqrt{\frac{F_T}{\mu}}$$

Musical Instruments

Standing waves in strings are the basis for the operation of stringed musical instruments. By plucking or hitting the string, a standing wave is formed in the rope since it is attached at each end. What mode will we have then? All possible modes can be present at the same time.
https://www.youtube.com/watch?v=PETuX_pXLNU

The sound coming from the instrument will therefore be the superposition of several sounds with frequencies corresponding to the frequencies of the possible harmonics. However, the note played by the instrument always corresponds to the frequency of the first harmonic. Thus, the note played by a rope will be

$$f_{note} = f_1 = \frac{1}{2L}\sqrt{\frac{F_T}{\mu}}$$

This equation represents very well what is happening with the strings of a guitar. (Note that high frequencies correspond to high-pitch tones and that low frequencies correspond to low-pitch sounds as we shall see in the next chapter.)

If you pluck a string, a sound is created. If the length of the string is reduced, by pressing a fret, the sound is more high-pitched. The formula does indeed tell us that f increases if L decreases.

If you pluck the largest string, the sound has a lower pitch than with the smallest string. The big string has a larger linear density than the little string (it is more massive for the same length). The formula also tells us that the frequency is lower if the linear density is larger.

If the string is stretched by turning the tuning peg, the tension increases. Then a higher-pitched sound is obtained. This time, the formula indicates that the frequency increases if the tension increases. The rope can also be stretched with a slight sideways motion while pressing the fret. A vibrato effect can be obtained by changing the tension in this way.

(The following paragraph is for those having a more advanced knowledge in music.)
The formula also indicates that the distance between the frets will be smaller and smaller as the length of the string is decreased. Suppose that the length of the rope has to be reduced from 0.6 m to 0.3 m to change the pitch from 300 Hz to 600 Hz (one octave). Then there are 12 semitones on a distance of 30 cm. To go one octave higher (from 600 Hz to 1200 Hz), the length of the rope must be reduced from 0.3 m to 0.15 m as the frequency is proportional to $1/L$. There are now 12 semitones on a distance of 15 cm. They are, therefore, closer to each other since the same number of semitones are located on a distance twice as short.

Mr. Borowicz makes a beautiful demonstration of all these effects in the first 5 minutes of this video.
http://www.youtube.com/watch?v=tXwJnr56LFo

Example 5.6.1

The largest string of a guitar has a length of 92.9 cm and a mass of 5.58 g. Once on the guitar, there is 65.5 cm between the two points of attachment of the string.

a) What should be the tension of the string so that the frequency of the first harmonic is 82.4 Hz (which is the frequency that the largest string of a guitar should have)?

The tension can be found with the formula for the frequency of the first harmonic

$$f_1 = \frac{1}{2L}\sqrt{\frac{F_T}{\mu}}$$

if the linear density is known. As the linear density of the string is

$$\begin{aligned}\mu &= \frac{mass}{length}\\ &= \frac{0.00558kg}{0.929m}\\ &= 0.00601\tfrac{kg}{m}\end{aligned}$$

the tension is

$$f_1 = \frac{1}{2L}\sqrt{\frac{F_T}{\mu}}$$

$$82.4Hz = \frac{1}{2 \cdot 0.655m} \cdot \sqrt{\frac{F_T}{0.00601\frac{kg}{m}}}$$
$$F_T = 70.03N$$

b) What are the frequencies of the 2nd, 3rd, and 4th harmonics?

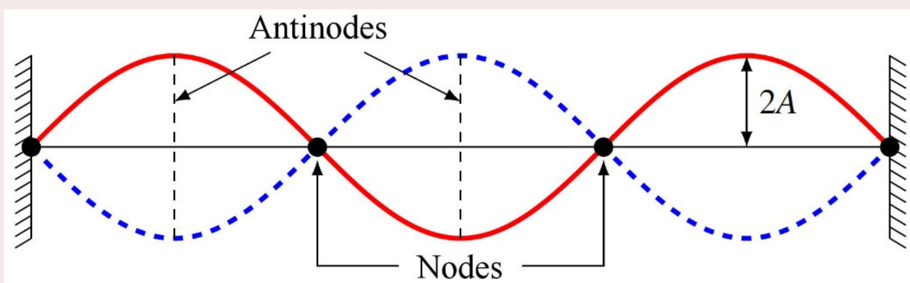
The frequency of the other harmonics is

$$f_n = nf_1$$
$$f_2 = 2 \cdot 82.4Hz = 164.8Hz$$
$$f_3 = 3 \cdot 82.4Hz = 247.2Hz$$
$$f_4 = 4 \cdot 82.4Hz = 329.6Hz$$

Each of these harmonics will be heard at the same time when the string is plucked.

c) If the amplitude of the oscillation of the standing wave is 0.1 mm at a position located 3 cm from the end of the string for the third harmonic, what is the amplitude of the oscillation in the centre of the antinode?

The amplitude at the centre of the antinode is 2*A* as seen previously.



To find this amplitude from the amplitude at another position, we will use the following formula.

$$A_{tot} = |2A\sin kx|$$

Knowing that A_{tot} = 0.1 mm at x = 0.03 m, we will be able to find 2*A* provided the value of k is known. k can be found with the wavelength. For the 3rd harmonic, the wavelength is

$$\lambda = \frac{2L}{3}$$
$$= \frac{2 \cdot 0.655m}{3}$$
$$= 0.4367m$$

The wave number is thus

$$k = \frac{2\pi}{\lambda}$$
$$= 14.39 \tfrac{rad}{m}$$

With an amplitude of 0.1 mm cm at x = 3 cm, we obtain

$$A_{tot} = \left|2A \sin kx\right|$$
$$0.0001m = \left|2A \cdot \sin\left(14.39 \tfrac{rad}{m} \cdot 0.03m\right)\right|$$
$$A = 0.0001195m$$

Therefore, the amplitude at the centre of the antinode is

$$2A = 0.000239m$$
$$= 0.239mm$$

Sometimes, you'll have the feeling that some information is missing in order to solve the problem, but it is not the case. By applying the same equation twice, the solution becomes simpler when one equation is divided by the other. Here is an example.

Example 5.6.2

A standing wave on a rope has a fundamental frequency of 100 Hz when the tension is 200 N. What should be the tension to have a fundamental frequency of 500 Hz?

We know that

$$f_1 = \frac{1}{2L}\sqrt{\frac{F_T}{\mu}}$$
$$100Hz = \frac{1}{2L}\sqrt{\frac{200N}{\mu}}$$

and we want to have

$$f_1' = \frac{1}{2L}\sqrt{\frac{F_T'}{\mu}}$$
$$500Hz = \frac{1}{2L}\sqrt{\frac{F_T'}{\mu}}$$

Dividing the second equation by the first, we arrive at

$$\frac{500Hz}{100Hz} = \frac{\frac{1}{2L}\sqrt{\frac{F_T'}{\mu}}}{\frac{1}{2L}\sqrt{\frac{200N}{\mu}}}$$

$$5 = \sqrt{\frac{F_T'}{200N}}$$
$$F_T' = 5000N$$

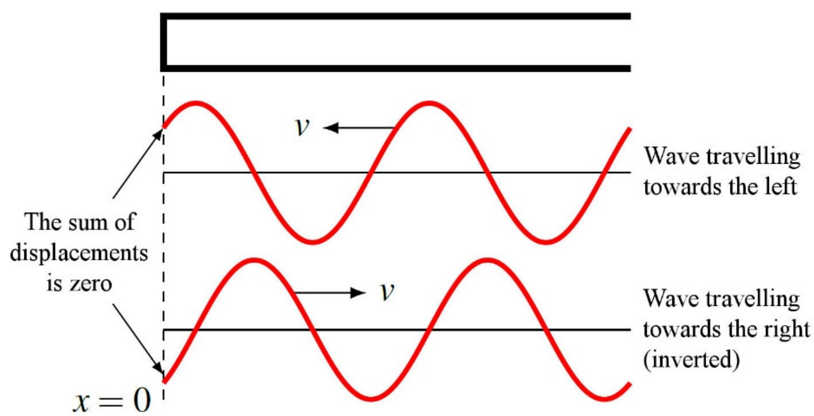
5.7 STANDING SOUND WAVES

Standing sound waves can be generated in a tube. To generate them, a sound wave is sent in a tube. When the wave reaches the end, it is reflected and goes back in the tube in the opposite direction. There are, therefore, two waves going in opposite directions, and a standing wave is created.

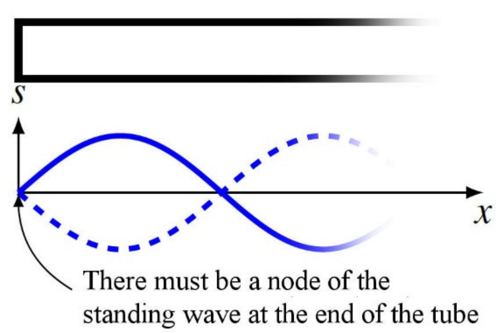
Reflection at the End of the Tube

Tube closed at the end

When a wave arrives at the closed end of a tube, the wave will obviously be reflected. A wave in air is then reflected on a solid wall whose impedance is much greater than air. Therefore, the wave will be inverted. (Beware, the graphs show the motion of air molecules, which is done horizontally, not vertically.)

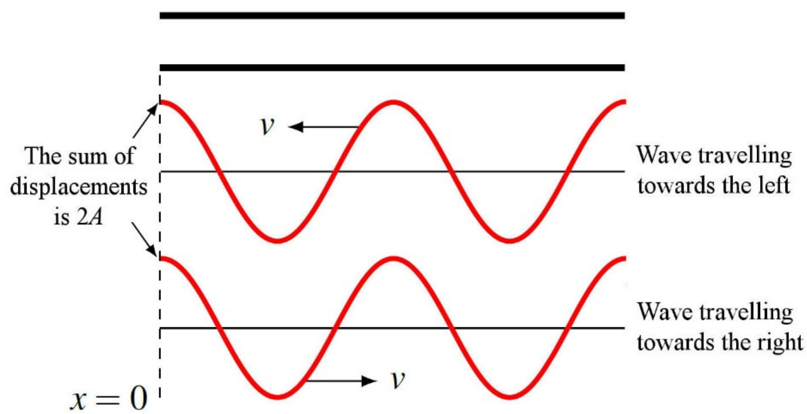


When the two waves are added, the result necessarily vanishes at the end of the tube because a displacement is added to another identical but opposed displacement at this place. A node of the stationary wave must then be located at the end of the tube wave.



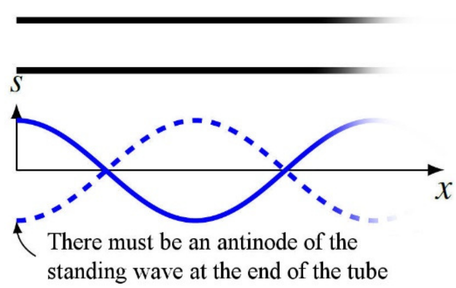
Tube opened at the end

When a wave reaches the open end of a tube, it can be thought at first glance that there will be no reflection. The wave is travelling in air inside the tube and outside the tube. Since there is no change of medium, the impedance remains the same and there should be no reflection. In reality, the situation is a bit more complex than that because the impedance also depends on the constraints acting on the medium. The impedance of the air inside a tube is greater than the impedance of the air outside the tube because the air inside the tube cannot expand in every direction. Thus, when the sound arrives at the open end of a tube, there is indeed a change of impedance and, therefore, part of the wave is reflected. Since the wave is reflected on a lower impedance medium, the reflected wave is not inverted.



When the two waves are added, the result is necessarily large at the end of the tube because two identical displacements are added at this place. When a crest is at the end of the tube (as can be seen in the diagram), another crest is added and the total displacement is $2A$. However, the only place where the amplitude of a standing wave is $2A$ is at the centre of the antinode of the standing wave. This means that the end of an open tube must correspond to the centre of an antinode of the standing wave.

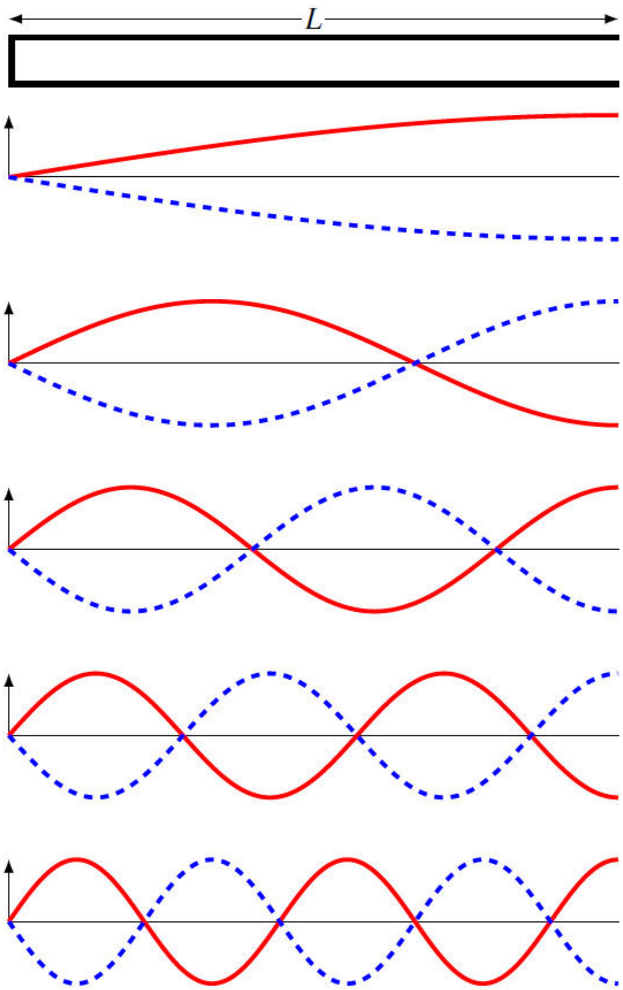
The centre of an antinode of the standing wave must, therefore, be located at the end of the tube. (Actually, the centre of the antinode is a little outside the tube, at a distance of 0.61 times the diameter of the tube.)



Possible Wavelengths and Frequencies

Open-closed Tube

If a tube is open at one end, the centre of an antinode must be located at this end of the tube. If the tube is closed at the other end, a node must be located at this end of the tube. Despite these constraints, there are several possible standing waves, called harmonics or modes.



$n = 1$ 1st mode = 1st harmonic $L = \dfrac{\lambda}{4}$

$n = 3$ 2nd mode = 3rd harmonic $L = \dfrac{3\lambda}{4}$

$n = 5$ 3rd mode = 5th harmonic $L = \dfrac{5\lambda}{4}$

$n = 7$ 4th mode = 7th harmonic $L = \dfrac{7\lambda}{4}$

$n = 9$ 5th mode = 9th harmonic $L = \dfrac{9\lambda}{4}$

The following link then exists between the length of tube *L* and the wavelength.

$$L = \frac{n\lambda_n}{4} \quad \text{where } n \text{ is an odd integer}$$

This odd integer is the harmonic number. Thus, the 4th mode corresponds to the 7th harmonic. The first mode can also be called the *fundamental mode*. Therefore,

Possible Wavelengths for a Stationary Sound Wave in a Closed Tube

$$\lambda_n = \frac{4L}{n} \quad \text{where } n \text{ is an odd integer}$$

The possible frequencies are obtained with $v = \lambda f$. Using the values of the possible wavelength, the frequencies are

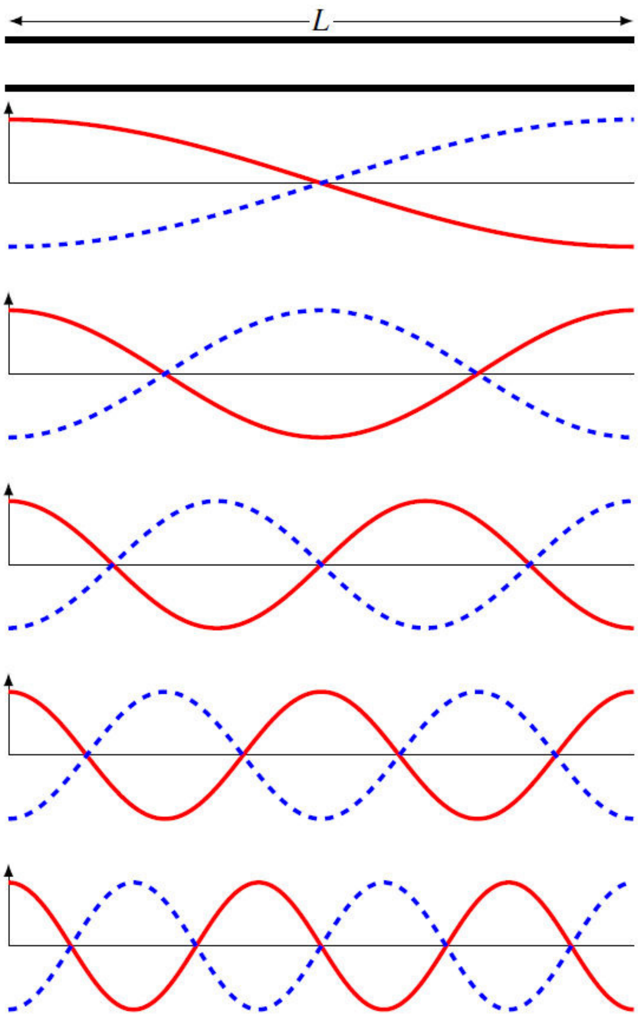
Possible Frequencies for a Stationary Sound Wave in a Closed Tube

$$f_n = \frac{nv}{4L} = nf_1 \quad \text{where } n \text{ is an odd integer}$$

The following animation shows the motion of the air molecules in a closed tube.
https://youtu.be/JfvTXmIFMKY

Open-open Tube

With a tube open at both ends, the centre of an antinode must be located at each end of the tube. Despite these constraints, there are several possible modes.



$n = 1$ 1st mode = 1st harmonic $L = \frac{\lambda}{2}$

$n = 2$ 2nd mode = 2nd harmonic $L = \lambda$

$n = 3$ 3rd mode = 3rd harmonic $L = \frac{3\lambda}{2}$

$n = 4$ 4th mode = 4th harmonic $L = 2\lambda$

$n = 5$ 5th mode = 5th harmonic $L = \frac{5\lambda}{2}$

The following link thus exists between the length of the tube L and the wavelength.

$$L = \frac{n\lambda_n}{2} \quad \text{where } n \text{ is an integer}$$

Again, n is the harmonic number. In this case, there's no difference between the mode and the harmonic number. The 4th mode corresponds to the 4th harmonic. Again, the first mode can be called the fundamental mode. The possible wavelengths are then

Possible Wavelengths for a Stationary Sound Wave in an Open Tube

$$\lambda_n = \frac{2L}{n} \text{ where } n \text{ is an integer}$$

The possible frequencies are obtained with $v = \lambda f$. Using the values of the possible wavelength, the frequencies are

Possible Frequencies for a Stationary Sound Wave in an Open Tube

$$f_n = \frac{nv}{2L} = nf_1 \text{ where } n \text{ is an integer}$$

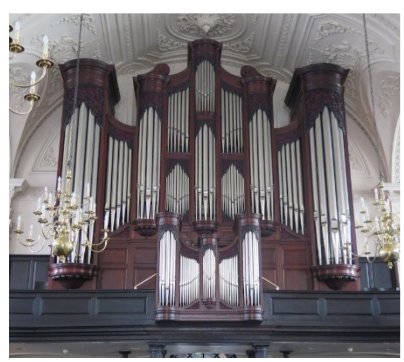
The case where the tube is closed at both ends will not be considered because the sound cannot enter the tube and form a standing wave (although that would be possible if a small speaker is inserted inside the tube). If we were to do it, we would then realize that the wavelength and the frequency, in this case, are exactly the same as for the tube open at both ends.

The following animation shows the motion of the air molecules in an open tube.
https://youtu.be/x0HyBi-VSwM

Musical Instruments

Standing waves in tubes generate the sound produced by wind musical instruments. For example, a recorder is simply a tube open at both ends. On one side, there is a whistle that serves as a sound source. The sound then enters the tube where it forms a standing wave. Which mode will be present then? All possible modes can be present at the same time. The sound heard is simply the superposition of multiple sounds with frequencies corresponding to the frequencies of the possible harmonics. The note played by the instrument is always the fundamental frequency (frequency of the fundamental mode or the first harmonic).

If the length of the tube is changed, the frequencies change. The formulas of the frequencies clearly indicate that the frequencies decrease if the length of the tube is increased. The sound then has a lower pitch. For a recorder or a flute, the length is changed by plugging holes with fingers. For an organ, the note is changed by producing sound in a tube with a different length. That is why an organ has so many pipes, as this organ in St-Martin-in-the-Fields in London.



Example 5.7.1

An organ pipe has a length of 26 cm, and the temperature of the air inside the pipe is 25 °C.

a) What are the frequencies of the first three modes if the pipe is open?

The frequencies of the harmonics of an open tube are given by

$$f_n = \frac{nv}{2L}$$

To calculate the frequencies, the speed of sound is needed. This speed is

$$\begin{aligned} v &= 331.3\tfrac{m}{s} \cdot \sqrt{\frac{298.15K}{273.15K}} \\ &= 346.1\tfrac{m}{s} \end{aligned}$$

Therefore, the fundamental frequency is

$$\begin{aligned} f_1 &= \frac{v}{2L} \\ &= \frac{346.1\tfrac{m}{s}}{2 \cdot 0.26m} \\ &= 666Hz \end{aligned}$$

The other frequencies are

$$\begin{aligned} f_2 &= 2f_1 = 1332Hz \\ f_3 &= 3f_1 = 1998Hz \end{aligned}$$

b) What are the frequencies of the first three modes if the pipe is closed?

The frequencies of the harmonics of a closed tube are given by

$$f_n = \frac{nv}{4L}$$

Thus, the fundamental frequency is

$$\begin{aligned} f_1 &= \frac{v}{4L} \\ &= \frac{346.1\tfrac{m}{s}}{4 \cdot 0.26m} \\ &= 333Hz \end{aligned}$$

The other frequencies are

$$f_3 = 3f_1 = 999Hz$$
$$f_5 = 5f_1 = 1665Hz$$

(The values of *n* must be odd in this case.)

Several wind instruments have a flared end. The trumpet is a great example.

/fr.vikidia.org/wiki/Trompette

This shape is needed so that the change of impedance at the end of the trumpet is done gradually. With a gradual change, the wave is not reflected back in the trumpet when the sound gets out of the instrument. To completely avoid reflections, the exact shape must be an exponential function.

Without thus flared shape, two standing waves would form in the instrument. For example, with a clarinet, there would be one wave between the mouth and the open hole, and another between the open hole and the end of the instrument. With a flared shape, there is no reflection and the end of the instrument, and the second wave does not form.

5.8 COMPLEX WAVES

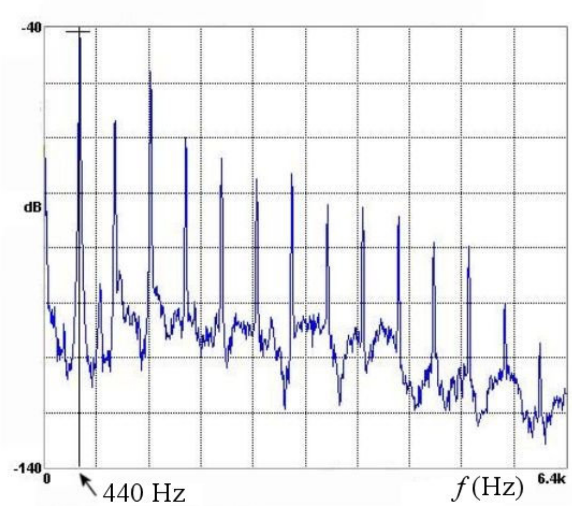
Why do a guitar and a piano sound different when they play the same note? If they play an A at 440 Hz, the waves should be identical. There is nothing that looks more like a sine wave than another sine wave of the same frequency.

The Intensity of the Harmonics

The sound is different because the sound of an instrument is not a simple sine wave. It is the result of the superimposition of several sine waves. More precisely, it is the result of the superimposition of different harmonics created by the instrument.

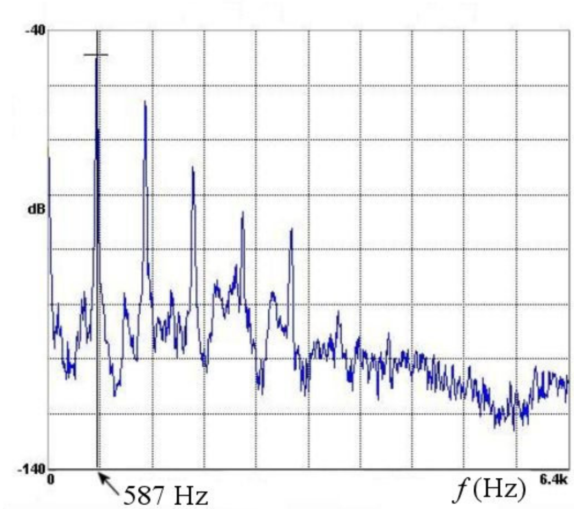
When an instrument is played at a certain note, the note matches the frequency of the first harmonic. However, this sound with this frequency is not the only sound generated. All the other harmonics are produced at the same time. (Harmonics are also called overtones or partials since harmonics are a specific type of overtones.) The relative intensity of these different harmonics can be, however, very different according to the instrument played. For example, here is a graph showing the intensity of the sound as a function of the frequency for a clarinet playing a 440 Hz note.

It can be seen that the intensity is great at 440 Hz, which is normal since this is the note played by the instrument. It can also be noted that there are peaks of intensity for all integer multiples of this frequency: these are the harmonics. For a clarinet at 440 Hz, it can be seen that the third harmonic (1320 Hz) is louder than the second harmonic (880 Hz). The intensity then gradually decreases for the other harmonics. These different harmonics played with these relative intensities create a clarinet sound. If the relative intensity of the harmonics is changed, the result does not sound like a clarinet.



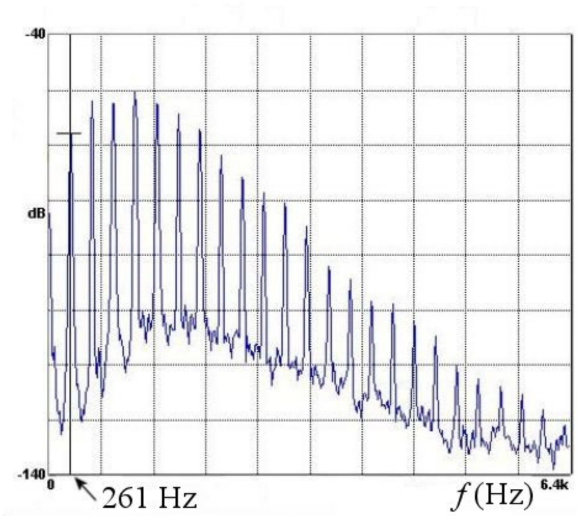
www.ugcs.caltech.edu/~tasha/

Now, here is the graph of the intensity of the sound as a function of the frequency for a flute at 587 Hz.



Again, it can be noted that the sound is composed of many harmonics. In this case, the relative intensity of the harmonics is not the same as for a clarinet. A sound composed of these harmonics with the intensities shown in the graph corresponds to the sound of a flute playing a note at 587 Hz.

The intensities of the harmonics are also different for the trumpet at 267 Hz.

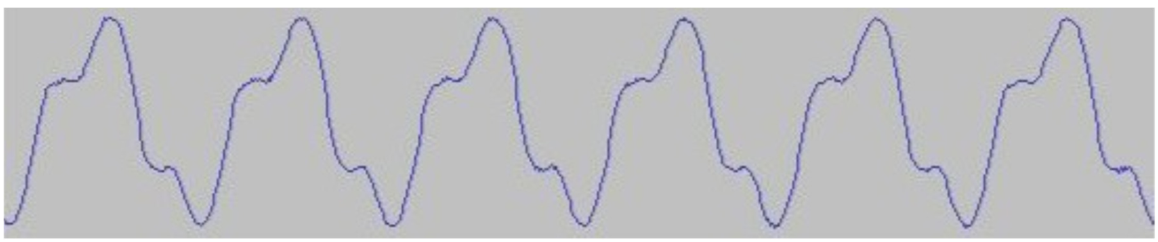


Obviously, the intensity of the harmonics is different here since the 4th harmonic is the one with the largest intensity after the fundamental frequency. A sound with harmonics having these intensities creates the sound of a trumpet.

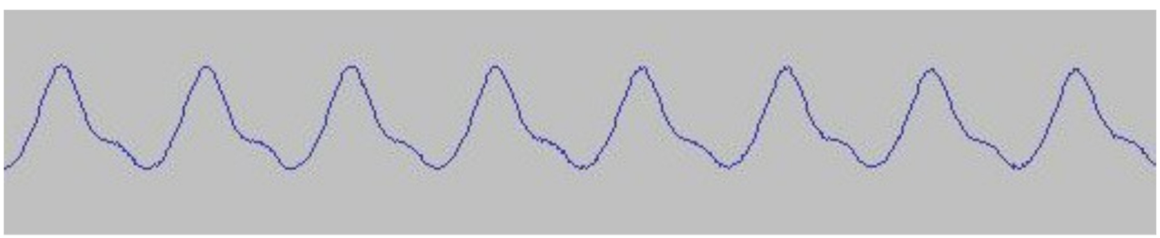
The Shape of the Wave

In fact, by adding several harmonics, a wave with a shape different from a sine function is created. It has been shown in the 19th century that any periodic wave is actually a superposition of sine waves whose frequency are integer multiples of the fundamental frequency (these are the harmonics). You can also have fun adding harmonics of variable intensity to see the shape and ear the resulting sound of the resulting wave.
https://www.compadre.org/osp/EJSS/4116/144.htm
This means that there is another way to understand why a guitar and a piano have a different sound. If the shape of the wave is different, then this means that the relative intensity of the different harmonics is different for each instrument.

The waveform of the sound made by a clarinet and quite different from the waveform of the sound made by a flute or a trumpet.



Clarinet at 440 Hz



Flute at 587 Hz

Trumpet at 261 Hz

www.ugcs.caltech.edu/~tasha/

In these three cases, the motion of air molecules is not given by a sine function, but by a much more complex function.

With a different waveform, the sound is different even if the frequency is the same.

Timbre

The difference between sounds of the same frequency is called ***timbre***. If a flute and clarinet have different sounds when they play the same note, it is because they have different timbre.

Timbre

Two instruments playing the same note have a different timbre because

1) The waveform is different
or (equivalent)
2) The relative intensity of the harmonics is different.

These two explanations are equivalent because the shape of the wave is changed if the relative intensity of the harmonics is changed.

Some Demonstrations

In the following demonstration, the sound of a single harmonic is played just before the sound of the instrument. It begins with the sound of the first harmonic, followed by the sound of the instrument. Then you'll hear the sound of the second harmonic followed the sound of the instrument and so on for all the harmonics. By doing this, it is easier to perceive the sound of a specific harmonic amid the sound of the instrument.
http://physique.merici.ca/ondes/sons/harmoniqueavantson.mp3

In this other demonstration, the harmonics are added one by one to form the sound of the instrument. It begins with the first harmonic only. The sound made by the first two harmonics follows, then the sound made by the first three harmonics and so forth. You'll see that several harmonics are needed before you can recognize the instrument.
http://physique.merici.ca/ondes/sons/ajoutharmonique.mp3

Actually, the relative intensity of the harmonics does not only varies for each instrument, it also changes for every note played by the instruments. The relative intensity of the harmonics of a bassoon playing the note A is not the same as for the same bassoon playing the note D. In the following demonstration, you will hear a bassoon. Initially, the bassoon plays several notes. Next, the sound of the instrument is changed so that the relative intensity of the harmonics is always the same as for the highest note played by the bassoon. Obviously, high-pitch notes have their normal sound, because the relative intensity has not changed for these notes. However, the low-pitch notes don't sound anymore as they should. This means that they do not have the same harmonic relative intensity as the high-pitch notes.
http://physique.merici.ca/ondes/sons/memespectreauxnotes.mp3

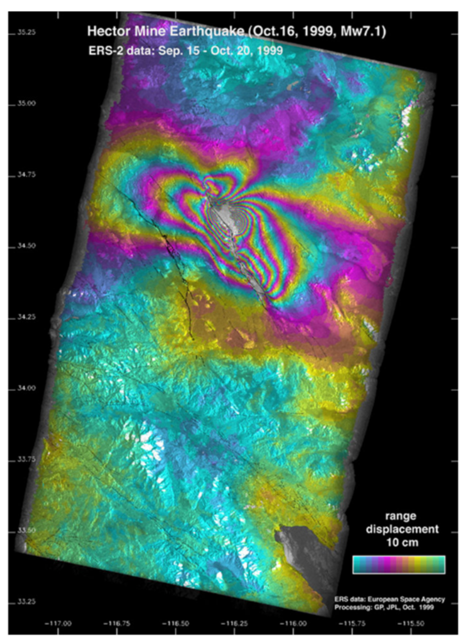
It is even more complicated than that because the variations of intensity over time are also part of the process to recognize an instrument. In the following demonstration, a Bach piece is played on the piano. Then, the same piece is played backwards (from end to beginning) and the recording of this piece is played in reverse, so that the original piece is heard again, except that each note in reversed in time. The relative intensity of the harmonics has remained the same since the same notes are played with the same instrument, but it does not sound like a piano…
http://physique.merici.ca/ondes/sons/attack-decay.mp3

Human voices have different timbres for the same reason: the relative intensity of the harmonics is different from one person to another. The sound is formed in the throat which plays the role of the tube. Standing waves form in the tube and several harmonics are present at the same time. According to the intensity of each harmonic, our voice has a specific timbre. By the way, the frequencies generated by your voice can be changed. If you breathe helium, the frequencies produced by your voice will change, not because the wavelengths have changed (they depend on the shape of the throat), but because the speed of sound in helium is different from the speed in the air. With a greater speed, the sounds generated have a higher frequency, and this gives you a higher-pitch voice. If you breathe sulphur hexafluoride (which is not toxic), the opposite will occur because the speed of sound in this gas is smaller than in air. The sound of your voice will then have a lower pitch.
http://www.youtube.com/watch?v=FvvSIAqOkIw

5.9 SOME APPLICATIONS OF INTERFERENCE

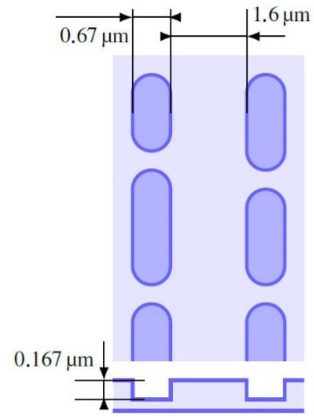
Interferometry

Here is another very interesting application of interference: some satellites send microwaves to the Earth's surface and measure the phase of the reflected wave. If the satellite returns above the same place after an earthquake that caused a change in ground elevation, the signal will take slightly less or slightly more time to come back to the satellite. The phase of the received signal will then change. By comparing the signals before and after the motion of the ground, interference is obtained and this interference allows geologists to measure the elevation variations of the ground. The image to the right shows what was obtained after an earthquake in California in 1999.

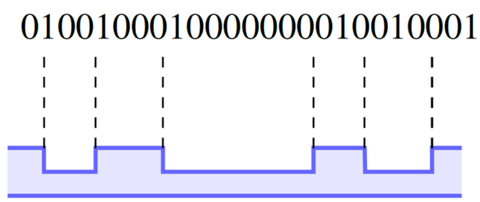


www-radar.jpl.nasa.gov/sect323/InSar4crust/HME/

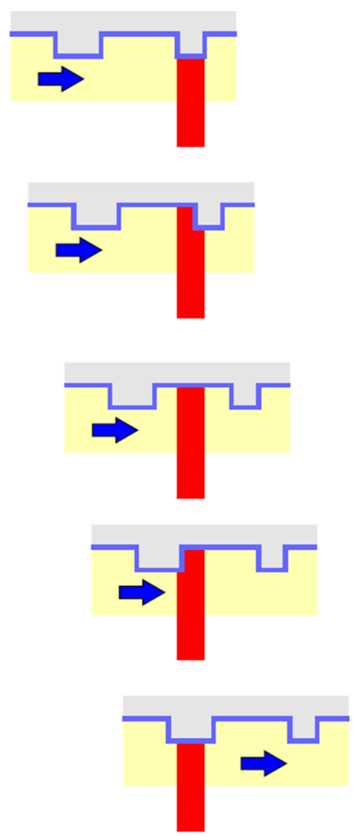
Reading a Compact Disc



Interference is also used to read a compact disc. On the surface of the disc, the information is coded with holes carved on the surface of the disc. The depth of the holes is equal to one-quarter of the wavelength of the laser used to read the disc. The untouched surface is called the *plateau*, and the holes are called *pits*.



The only data written on the disc are 0 and 1. For a 0, the surface of the disc stays the same. For a 1, there must be a transition from a pit to the plateau, or from the plateau to a pit, as shown in the diagram.



The reading is done with a laser. (You won't see the laser in your player since it has a 780 nm wavelength in air, which is in the infrared part of the spectrum.) When the laser strikes a uniform surface (1st, 3rd and 5th images), it is reflected normally and the sensor captures the reflection. When the laser comes to a place where the height changes (2nd and 4th images), a part of the laser is reflected on the plateau and another part is reflected in the bottom of a pit. However, as the depth of the hole is $\lambda/4$, the extra distance travelled by the part of the laser beam reflected on the bottom of the pit is $\lambda/2$ (round trip), which corresponds to a phase shift of π between the two parts of the laser beam. Therefore, there is destructive interference between the two parts of the laser beam and the sensor receives nothing. So, when the sensor receives light, it's a 0 and when it receives nothing, it's a 1.

(You may notice that the depth of the pits (167 nm) is not equal to one-quarter of the wavelength of the laser (780 nm). But the extra distance is travelled in polycarbonate, and the wavelength is then divided by the refractive index of polycarbonate.)

Noise-cancelling Headphones

Headphones which eliminate ambient noise can be bought. On these headphones, there are microphones which detect ambient noise. The headphones then emit an identical wave, but out of phase of π radians called the *anti-noise*. These two waves then make destructive interference, thereby eliminating ambient noise. This is called *active noise control*.

SUMMARY OF EQUATIONS

Superposition Principle

$$y_{tot} = y_1 + y_2 + y_3 + \dots$$

Resulting Amplitude of Two Interfering Waves With the Same Amplitude *A*

$$A_2 = \left| 2A_1 \cos\left(\frac{\Delta\phi}{2}\right)\right|$$

Resulting Amplitude of Two Interfering Waves With Different Amplitude A_1 and A_2

$$A_{tot} = \sqrt{A_1^2 + 2A_1A_2\cos(\Delta\phi) + A_2^2}$$

Condition to Obtain Constructive Interference

$$\Delta\phi = 2m\pi \text{ where } m \text{ is an integer}$$

Amplitude When There Is Constructive Interference

$$A_{tot} = A_1 + A_2$$

Condition to Obtain Destructive Interference

$$\Delta\phi = (2m+1)\pi \text{ where } m \text{ is an integer}$$

Amplitude When There Is Destructive Interference

$$A_{tot} = |A_1 - A_2|$$

Phase Difference Between Two Waves

$$\Delta\phi = \Delta\phi_T + \Delta\phi_S + \Delta\phi_R$$

$\Delta\phi_T$ Calculation

$$\Delta\phi_T = -\omega\Delta t = -\frac{\Delta t}{T}2\pi$$

$$\Delta\phi_T = -\frac{\Delta r}{\lambda}2\pi$$

$\Delta\phi_S$ Calculation

$$\Delta\phi_S = \phi_{source\ 2} - \phi_{source\ 1}$$

$\Delta\phi_R$ Calculation

$$\phi_R = 0 \text{ (not inverted) or } \pi \text{ (inverted)}$$
$$\Delta\phi_R = \phi_{R2} - \phi_{R1}$$

Phase Difference Between the Two Waves Reflected in Thin Films

$$\Delta\phi = \frac{4\pi n_f e}{\lambda_0} + \Delta\phi_R$$

where $\Delta\phi_R$ can only be 0, -π, or π depending on the refractive indices.

Frequency of the Oscillation With Two Waves of Different Frequencies

$$f_{osc} = \frac{f_1 + f_2}{2}$$

Frequency of the Changes of Amplitude

$$f_{ampli} = |f_1 - f_2|$$

Beats

$$f_{sound} = \frac{f_1 + f_2}{2} \qquad f_{beats} = |f_1 - f_2|$$

Standing Wave Equation

$$y_{tot} = 2A\sin kx \cos\omega t$$

Oscillation Amplitude for a Standing Wave

$$A_{tot} = |2A\sin kx|$$

Possible Wavelengths of a Standing Wave on a Rope

$$\lambda_n = \frac{2L}{n} \text{ (n is a positive integer)}$$

Distance between the Nodes of a Standing Wave

$$\text{distance} = \frac{\lambda}{2}$$

Possible Frequencies of a Standing Wave on a Rope

$$f_n = \frac{nv}{2L} = nf_1$$

Possible Frequencies of a Standing Wave on a Rope

$$f_n = \frac{n}{2L}\sqrt{\frac{F_T}{\mu}}$$

Possible Wavelengths for a Stationary Sound Wave in a Closed Tube

$$\lambda_n = \frac{4L}{n} \text{ where } n \text{ is an odd integer}$$

Possible Frequencies for a Stationary Sound Wave in a Closed Tube

$$f_n = \frac{nv}{4L} = nf_1 \quad \text{where } n \text{ is an odd integer}$$

Possible Wavelengths for a Stationary Sound Wave in an Open Tube

$$\lambda_n = \frac{2L}{n} \text{ where } n \text{ is an integer}$$

Possible Frequencies for a Stationary Sound Wave in an Open Tube

$$f_n = \frac{nv}{2L} = nf_1 \quad \text{where } n \text{ is an integer}$$

Timbre

Two instruments playing the same note have a different timbre because

1) The waveform is different

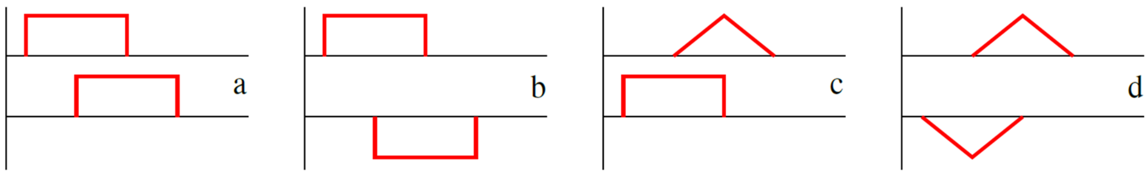
or (equivalent)

2) The relative intensity of the harmonics is different.

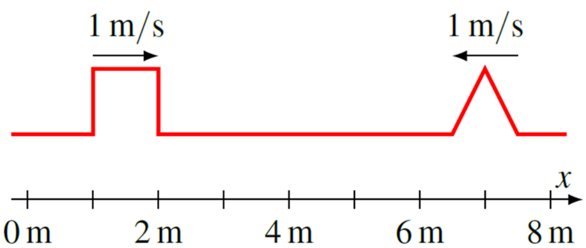
EXERCISES

5.1 Superposition Principle

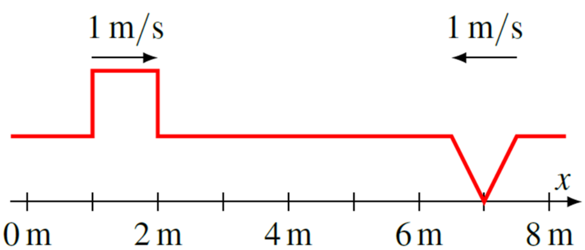
1. Draw the result of the superposition of these waves.



2. Draw the resulting wave 2.5 seconds later.



3. Draw the resulting wave 2.5 seconds later.



5.2 Superposition of Two Sine Waves

4. What is the amplitude of the resulting wave when the two following waves are added?

$$y_1 = 0.2m \cdot \sin\left(20\tfrac{rad}{m} \cdot x + 45\tfrac{rad}{s} \cdot t + 4rad\right)$$
$$y_2 = 0.2m \cdot \sin\left(20\tfrac{rad}{m} \cdot x + 45\tfrac{rad}{s} \cdot t - 1rad\right)$$

5. What is the amplitude of the resulting wave when the two following waves are added?

$$y_1 = 0.5m \cdot \sin\left(10\tfrac{rad}{m} \cdot x - 60\tfrac{rad}{s} \cdot t + 2rad\right)$$
$$y_2 = 0.4m \cdot \sin\left(10\tfrac{rad}{m} \cdot x - 60\tfrac{rad}{s} \cdot t - 1.5rad\right)$$

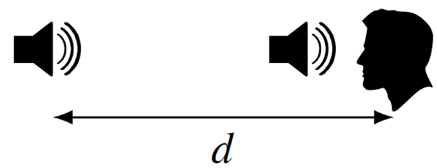
6. What wave (with the same amplitude) should be added to the wave $y_1 = 0.1m \cdot \sin\left(30\frac{rad}{m} \cdot x + 100\frac{rad}{s} \cdot t + 1rad\right)$ to obtain…

   a) constructive interference?
   b) destructive interference?

   (Give answers with a phase constant between 0 and 2π.)

5.3 Phase Difference Between Two Waves

Use this diagram to all the questions in this section.



7. Two sources emit (in phase) sound waves having a 50 cm wavelength. An observer is 5.2 m away from source A and 3.6 m away from source B. What is the phase difference between the waves received by the observer from these sources?

8. Two sources emit sound waves having a 40 cm wavelength. The sources are equidistant from an observer. What is the phase difference between the waves received by the observer from these sources knowing that source B is ahead of source A by a third of a cycle?

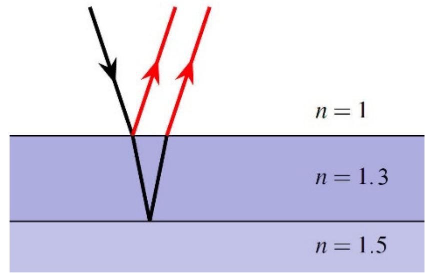
9. Two sources emit sound waves having a 20 cm wavelength. An observer is 5 m away from source A and 3 m away from source B. What is the phase difference between the waves received by the observer from these sources knowing that source B is lagging behind source A by one-quarter of a cycle?

10. Two in-phase loudspeakers both emit identical sound waves with a 25 cm wavelength. What is the smallest distance between the speakers (d) that will produce destructive interference for a listener standing in front of them?

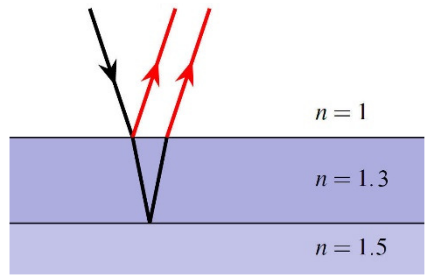
11. There are two speakers (A and B) which are both emitting 32 cm wavelength sound waves. However, speaker A lags behind speaker B by one-eighth of a cycle. Hildegard hears the sound coming from these two speakers. What is the smallest path length difference (Δr) that will produce destructive interference at the place where Hildegard is?

5.4 Thin Films

12. In the situation shown in the diagram, what are the wavelengths having a weak intensity (thus making destructive interference) in the reflected light (in the visible part of the spectrum) if the film has a 450 nm thickness?



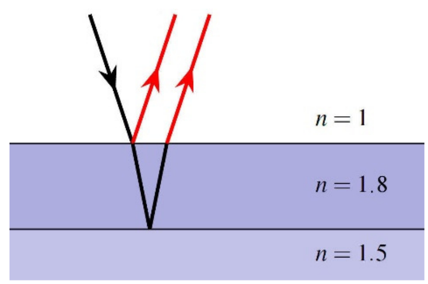
13. In the situation shown in the diagram, what are the wavelengths having a large intensity (thus making constructive interference) in the reflected light (in the visible part of the spectrum) if the film has a 450 nm thickness?



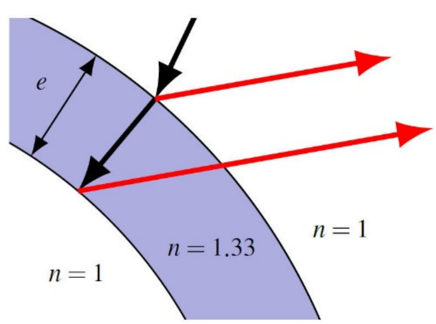
14. A thin film of oil having a 250 nm thickness and a 1.6 refractive index floats on the surface of a lake (n = 1.33). Blue light, whose wavelength is 450 nm, is reflected on both surfaces of this film.

   a) What is the phase difference between the two reflected waves?
   b) What is the amplitude of the reflected light compared to the amplitude there would be if there were no oil on the lake if it is assumed that the amplitudes of the reflected waves are the same?

15. In the following situation, a light wave, whose wavelength is 550 nm, is reflected and make constructive interference. What is the minimum possible thickness of the film?



16. White light is reflected on a soap bubble. In the reflected visible light, light having a 478.8 nm wavelength is making destructive interference while light having a 638.4 nm wavelength is making constructive interference. What is the minimum possible thickness of the wall of the bubble?



5.5 Superposition of Two Waves With Different Frequencies

17. Two sounds with frequencies of 500 Hz and 508 Hz are played simultaneously.

 a) What is the frequency of the sound heard?
 b) What is the frequency of the beats?

18. Two sine sound waves reach a person simultaneously. The person then hears a sound with a frequency of 350 Hz and beats with a 6 Hz frequency. What are the frequencies of the two sine waves?

19. To tune a guitar, you can use a device (or an application on the iPad) that produces a sound with a frequency that the guitar string must have. Pierre-Paul uses such a device to tune the smallest guitar string. The device thus produces a 329.6 Hz sound, which is the frequency that this string must have when played. By playing the device and the string at the same time, Pierre-Paul hears beats having a frequency of 4.2 Hz. At this time, the tension of the rope is 1300 N. He also noted that if he increases the tension of the string, the beat frequency decreases. What should be the tension of the string so that it produces a 329.6 Hz sound?

20. Using a police radar gun operating with a 25 GHz frequency, the frequency of the change of amplitude obtained by combining the emitted and reflected waves is 6000 Hz. What is the speed of the car?

5.6 Standing Waves

21. The equation of a standing wave is

$$y_{tot} = 6cm \cdot \sin\left(40\pi \tfrac{rad}{m} \cdot x\right) \cdot \cos\left(200\pi \tfrac{rad}{s} \cdot t\right)$$

 a) What is the frequency of this wave?
 b) What is the wavelength of the wave?
 c) What is the velocity of the waves on this rope?
 d) What is the velocity of the rope at $x = 0.02$ m and $t = 0.022$ s?
 e) What is the amplitude of the oscillations at $x = 0.5$ cm?

22. These two waves overlap to form a standing wave.

$$y_1 = 2cm \cdot \sin\left(10\pi \tfrac{rad}{m} \cdot x - 50\pi \tfrac{rad}{s} \cdot t\right)$$
$$y_2 = 2cm \cdot \sin\left(10\pi \tfrac{rad}{m} \cdot x + 50\pi \tfrac{rad}{s} \cdot t\right)$$

What is the equation of the resulting standing wave $y_{tot} = 2A\sin(kx)\cos(\omega t)$?

23. Two sine waves going in opposite directions meet to form a standing wave on a string. Both waves have an amplitude of 5 cm, a wavelength of 20 cm and a period of 0.05 s.

a) What is the equation of the standing wave?
$$y_{tot} = 2A\sin(kx)\cos(\omega t)$$
b) What is the distance between the nodes of the standing wave?
c) What is the amplitude of the oscillation 1 cm from a node?

24. A standing wave has an amplitude of 5 mm at the centre of the antinodes, and the distance between the nodes of the standing wave is 20 cm. The tension of the rope is 12 N, and the linear density of the string is 30 g/m. What is the equation of the standing wave?
$$y_{tot} = 2A \cdot \sin(kx)\cos(\omega t)$$

25. A string has a length of 60 cm and a linear density of 20 g/m. What should be the tension of the string so that the frequency of the fourth harmonic is 400 Hz?

26. The frequency of the second harmonic on a string is 400 Hz. What is the speed of the waves on this string if it has a length of 2 m?

27. The equation of a standing wave is

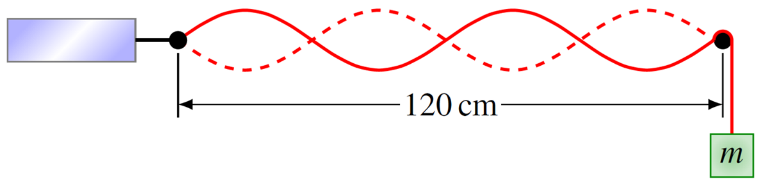
$$y_{tot} = 2cm \cdot \sin\left(20\pi \tfrac{rad}{m} \cdot x\right) \cdot \cos\left(300\pi \tfrac{rad}{s} \cdot t\right)$$

What is the length of the string if this standing wave corresponds to the third harmonic?

28. A string has a length of 50 cm and a tension of 350 N. What is the linear density of the string if the fundamental frequency of the standing wave is 50 Hz?

29. A taut rope has a 200 Hz fundamental frequency if the tension is 100 N. What should be the tension so that the frequency of the fifth harmonic is 500 Hz?

30. What should be the mass of the hanging object to get this standing wave knowing that the linear density of the string is 36 g/m and that the oscillation frequency is 160 Hz?



31. A taut rope has a fundamental frequency of 50 Hz if the string has a length of 100 cm. What should be the length of a rope having the same tension and the same linear density so that the frequency of the second harmonic is 400 Hz?

32. String A has a length of 1 m and a tension of 100 N while the string B has a length of 25 cm and a tension of 25 N. What is the ratio of the fundamental frequency of string A and the fundamental frequency of string B (f_{1A}/f_{1B}) if the strings have the same linear density?

33. Rope A is stretched and has a fundamental frequency of 360 Hz. What will the fundamental frequency of rope B, which has the same tension and the same linear density as rope A, be if its length is only 40% of the length of rope A?

34. In a rope, the frequency of a harmonic is 520 Hz. The following harmonic has a frequency of 650 Hz. What is the frequency of the first harmonic?

5.7 Standing Sound Waves

35. A 25 cm long pipe is open at both ends. What is the frequency of the first three modes if the speed of sound is 340 m/s?

36. A 40 cm long pipe is open at one end and closed at the other end. What is the frequency of the first three modes if the speed of sound is 340 m/s?

37. A sound source sends a 500 Hz sound in a pipe. The other end of the pipe is closed. What should be the length of the pipe so that there is a standing wave corresponding to the third harmonic if the air temperature is 30 °C?

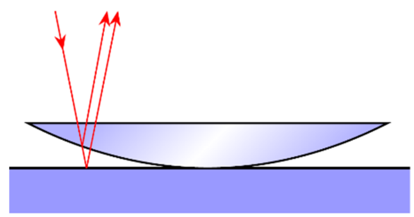
38. The frequency of the fundamental mode of a pipe is 500 Hz when the air temperature is 25 °C. What will the frequency of the fundamental mode be if the temperature of the air drops to 0 °C?

39. A sound is sent in a tube with an open end. Then, it can be noted that there is a standing wave at 630 Hz and that the frequency of the next mode is 840 Hz.

    a) Is the other end of the tube open or closed?
    b) What is the length of the tube if the speed of sound is 336 m/s?

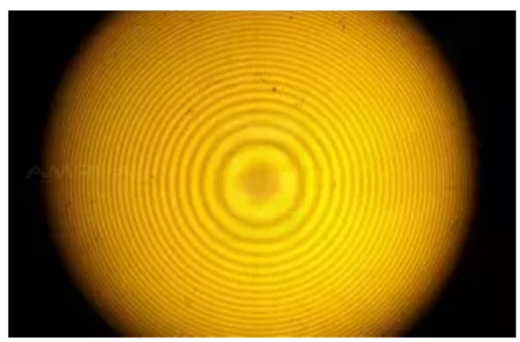
Challenges

(Questions more difficult than the exam questions.)



40. When a convergent lens made of glass having the shape shown on the diagram is placed on a glass surface, circular maxima and minima of interference are obtained.

    These circles are called Newton's rings. What is the radius of the 3rd dark ring if the radius of the curved face of the lens is 60 cm and if the wavelength of light is 600 nm? (Do not count the central dark spot as a ring.)

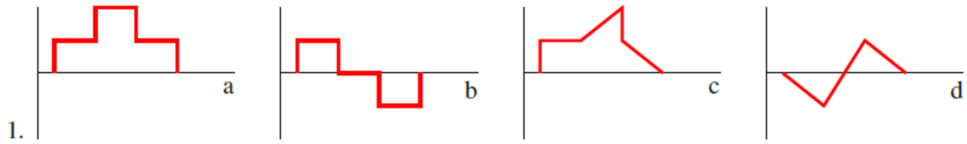


www.quora.com/What-is-a-Newton-ring

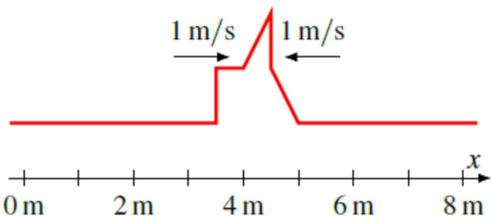
ANSWERS

5.1 Superposition Principle

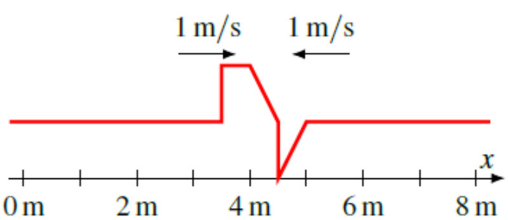
1.



2.



3.



5.2 Superposition of Two Sine Waves

4. 32.05 cm
5. 18.82 cm
6. a) $y_2 = 0.1m \cdot \sin\left(30\frac{rad}{m} \cdot x + 100\frac{rad}{s} \cdot t + 1\right)$

   b) $y_2 = 0.1m \cdot \sin\left(30\frac{rad}{m} \cdot x + 100\frac{rad}{s} \cdot t + 4{,}142\right)$

5.3 Phase Difference Between Two Waves

7. 32π/5
8. 2π/3
9. 61.26 rad
10. 12.5 cm
11. Speaker B must be 12 cm closer than speaker A.

5.4 Thin Films

12. Only the wavelength of 468 nm is absent.
13. Only the 585 nm wavelength is strongly reflected.

14. a) 14.31 rad b) The amplitude is 1.286 times greater than it would be without a thin film
15. 76.39 nm
16. 360 nm

5.5 Superposition of Two Waves With Different Frequencies

17. a) 504 Hz b) 8 Hz
18. 347 Hz and 353 Hz
19. 1333.8 N
20. 129.6 km/h

5.6 Standing Waves

21. a) 100 Hz b) 5 cm c) 5 m/s d) -21.07 m/s e) 3.527 cm
22. $y_{tot} = 4cm \cdot \sin\left(10\pi \frac{rad}{m} \cdot x\right) \cdot \cos\left(50\pi \frac{rad}{s} \cdot t\right)$
23. a) $y_{tot} = 0.1m \cdot \sin\left(10\pi \frac{rad}{m} \cdot x\right) \cdot \cos\left(40\pi \frac{rad}{s} \cdot t\right)$ b) 10 cm c) 3.090 cm
24. $y_{tot} = 0.005m \cdot \sin\left(5\pi \frac{rad}{m} \cdot x\right) \cdot \cos\left(100\pi \frac{rad}{s} \cdot t\right)$
25. 288 N
26. 800 m/s
27. 15 cm
28. 0.14 kg/m
29. 25 N
30. 33.85 kg
31. 25 cm
32. 0.5
33. 900 Hz
34. 130 Hz

5.7 Standing Sound Waves

35. 680 Hz 1360 Hz and 2040 Hz
36. 212.5 Hz 637.5 Hz and 1062.5 Hz
37. 52.35 cm
38. 478.6 Hz
39. a) open b) 0.8 m

Challenges

40. 1.039 mm