Chapter 1 Solutions

1. a) The amplitude is 20 cm.
b) The period is

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ &= \frac{2\pi}{5s^{-1}} \\ &= 1.257s \end{aligned}$$

c) The phase constant is π/4.
d) The maximum speed is

$$\begin{aligned} v_{\max} &= A\omega \\ &= 0.2m \cdot 5s^{-1} \\ &= 1\tfrac{m}{s} \end{aligned}$$

e) The equation is

$$x = 0.2m \cdot \cos\left(5\tfrac{rad}{s}t - \tfrac{\pi}{4}rad\right)$$

(The phase constant with a cosine function is π/2 smaller than with a sine function.)

2. a) The amplitude is 5 cm.
b) The period is 2.5 s.
c) This graph is the graph of a sine function shifted 0.4 s towards the left (approximately). Therefore, the phase constant is

$$\begin{aligned} \phi &= \frac{\Delta t}{T}2\pi \\ &= \frac{0.4s}{2.5s} \cdot 2\pi \\ &= 1.0053rad \end{aligned}$$

As this is an approximation, we could say that the phase constant is about 1 rad.

3. a) The position at t = 1 s is

$$\begin{aligned} x &= 0.25m \cdot \sin\left(10\tfrac{rad}{s} \cdot t + \tfrac{3\pi}{4} rad\right) \\ &= 0.25m \cdot \sin\left(10\tfrac{rad}{s} \cdot 1s + \tfrac{3\pi}{4} rad\right) \\ &= -0.05212m \end{aligned}$$

b) The velocity formula is

$$\begin{aligned} v &= A\omega\cos\left(\omega t + \phi\right) \\ &= 0.25m \cdot 10s^{-1} \cdot \cos\left(10\tfrac{rad}{s} \cdot t + \tfrac{3\pi}{4} rad\right) \\ &= 2.5\tfrac{m}{s} \cdot \cos\left(10\tfrac{rad}{s} \cdot t + \tfrac{3\pi}{4} rad\right) \end{aligned}$$

The velocity at t = 1 s is thus

$$\begin{aligned} v &= 2.5\tfrac{m}{s} \cdot \cos\left(10\tfrac{rad}{s} \cdot 1s + \tfrac{3\pi}{4} rad\right) \\ &= 2.445\tfrac{m}{s} \end{aligned}$$

c) The acceleration formula is

$$\begin{aligned} a &= -A\omega^2 \sin\left(\omega t + \phi\right) \\ &= -2.5\tfrac{m}{s} \cdot 10s^{-1} \cdot \sin\left(10\tfrac{rad}{s} \cdot t + \tfrac{3\pi}{4} rad\right) \\ &= -25\tfrac{m}{s^2} \cdot \sin\left(10\tfrac{rad}{s} \cdot t + \tfrac{3\pi}{4} rad\right) \end{aligned}$$

Therefore, the acceleration at t = 1 s is

$$\begin{aligned} a &= -25\tfrac{m}{s^2} \cdot \sin\left(10\tfrac{rad}{s} \cdot 1s + \tfrac{3\pi}{4} rad\right) \\ &= 5.216\tfrac{m}{s^2} \end{aligned}$$

4. The maximum speed is

$$v_{\max} = A\omega$$

ω can be found with the oscillation frequency.

$$\begin{aligned} \omega &= 2\pi f \\ &= 2\pi \cdot 5Hz \\ &= 10\pi\tfrac{rad}{s} \end{aligned}$$

The amplitude is then found with the maximum acceleration.

$$a_{\max} = A\omega^2$$
$$12\tfrac{m}{s^2} = A\cdot\left(10\pi s^{-1}\right)^2$$
$$A = 0.01216m$$

The maximum speed is thus

$$\begin{aligned} v_{\max} &= A\omega \\ &= 0.01216m\cdot 10\pi s^{-1} \\ &= 0.382\tfrac{m}{s} \end{aligned}$$

5. a) We have

$$v_{\max} = A\omega$$
$$a_{\max} = A\omega^2$$

Those give us the two following equations

$$32\tfrac{m}{s} = A\omega$$
$$128\tfrac{m}{s^2} = A\omega^2$$

that must be solved.

It is possible to solve for a variable in an equation and replace the other, but the solution is more easily obtained with a ratio.

$$\frac{A\omega^2}{A\omega} = \frac{128\tfrac{m}{s^2}}{32\tfrac{m}{s}}$$
$$\omega = 4\tfrac{rad}{s}$$

Therefore, the period is

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ &= \frac{2\pi}{4s^{-1}} \\ &= 1.571s \end{aligned}$$

b) The amplitude is

$$32\tfrac{m}{s} = A\omega$$
$$32\tfrac{m}{s} = A \cdot 4s^{-1}$$
$$A = 8m$$

6. In the formula

$$x = A\sin\left(\omega t + \phi\right)$$

we need to find A, ω and ϕ.

First, let's find the value of ω.

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{8s} = \tfrac{\pi}{4}\tfrac{rad}{s}$$

Then, the amplitude can be found.

$$A^2 = x^2 + \left(\frac{v}{\omega}\right)^2$$
$$= \left(0.1m\right)^2 + \left(\frac{0.24\tfrac{m}{s}}{\tfrac{\pi}{4}s^{-1}}\right)^2$$
$$= 0.10338m^2$$
$$A = 0.3215m$$

Finally, the phase constant is

$$\tan\left(\omega t + \phi\right) = \frac{x\omega}{v}$$
$$\tan\left(0 + \phi\right) = \frac{0.1m \cdot \tfrac{\pi}{4}s^{-1}}{0.24\tfrac{m}{s}}$$
$$\tan\left(\phi\right) = \tfrac{5\pi}{48}$$
$$\phi = 0.3163rad$$

Therefore, the equation is

$$x = 0.3215m \cdot \sin\left(\tfrac{\pi}{4}\tfrac{rad}{s} \cdot t + 0.3163rad\right)$$

7. In the formula

$$x = A\sin\left(\omega t + \phi\right)$$

we need to find *A*, ω and ϕ.

First, let's find the value of ω.

$$\begin{aligned}\omega &= \frac{2\pi}{T} \\ &= \frac{2\pi}{8s} \\ &= \tfrac{\pi}{4}\tfrac{rad}{s}\end{aligned}$$

Then, the amplitude can be found.

$$\begin{aligned}A^2 &= x^2 + \left(\frac{v}{\omega}\right)^2 \\ &= \left(-0.2m\right)^2 + \left(\frac{0\frac{m}{s}}{\frac{\pi}{4}s^{-1}}\right)^2 \\ &= 0.04m^2 \\ A &= 0.2m\end{aligned}$$

Finally, the phase constant is

$$\begin{aligned}\tan\left(\omega t + \phi\right) &= \frac{x\omega}{v} \\ \tan\left(0 + \phi\right) &= \frac{-0.2m \cdot \frac{\pi}{4}s^{-1}}{0\frac{m}{s}} \\ \tan\left(\phi\right) &= -\infty \\ \phi &= -\tfrac{\pi}{2}rad\end{aligned}$$

Therefore, the equation is

$$x = 0.2m \cdot \sin\left(\tfrac{\pi}{4}\tfrac{rad}{s} \cdot t - \tfrac{\pi}{2} rad\right)$$

8. a) The period will be found with

$$T = \frac{2\pi}{\omega}$$

ω is found with

$$\begin{aligned} a &= -\omega^2 x \\ -24\tfrac{m}{s^2} &= -\omega^2 \cdot 0.06m \\ \omega^2 &= 400\tfrac{rad^2}{s^2} \\ \omega &= 20\tfrac{rad}{s} \end{aligned}$$

Therefore, the period is

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ &= \frac{2\pi}{20s^{-1}} \\ &= 0.31416s \end{aligned}$$

b) The amplitude is

$$\begin{aligned} A^2 &= x^2 + \left(\frac{v}{\omega}\right)^2 \\ &= (0.06m)^2 + \left(\frac{-1\frac{m}{s}}{20s^{-1}}\right)^2 \\ &= 0.0061m^2 \\ A &= 0.0781m \end{aligned}$$

9. a) The velocity is found with

$$A^2 = x^2 + \left(\frac{v}{\omega}\right)^2$$

$$(0.25m)^2 = (0.15m)^2 + \left(\frac{v}{10s^{-1}}\right)^2$$

$$v = 2\tfrac{m}{s}$$

b) The acceleration is

$$\begin{aligned} a &= -\omega^2 x \\ &= -\left(10s^{-1}\right)^2 \cdot 0.15m \\ &= -15\tfrac{m}{s^2} \end{aligned}$$

10. a) When the object is at x = 12 cm, we have

$$0.12m = 0.2m \cdot \sin\left(5\tfrac{rad}{s} \cdot t + \tfrac{\pi}{4} rad\right)$$

$$0.6 = \sin\left(5\tfrac{rad}{s} \cdot t + \tfrac{\pi}{4} rad\right)$$

$$\begin{aligned} 0.6435 &= 5s^{-1} \cdot t + \tfrac{\pi}{4} \qquad & and \qquad & 2.4981 = 5s^{-1} \cdot t + \tfrac{\pi}{4} \\ -0.1419 &= 5s^{-1} \cdot t \qquad & and \qquad & 1.713 = 5s^{-1} \cdot t \\ t &= -0.02838s \qquad & and \qquad & t = 0.3425s \end{aligned}$$

The period can then be added (or removed) to each of these answers to find all the instants where the object is at x = 12 cm. The period is

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ &= \frac{2\pi}{5s^{-1}} \\ &= 1.2566s \end{aligned}$$

We thus have

$$\begin{array}{ccc} t=-0.02838s & & t=0.3425s \\ & \downarrow +1.2566s & \\ t=1.228s & & t=1.5991s \\ & \downarrow +1.2566s & \\ t=2.485s & & t=2.856s \\ & \downarrow +1.2566s & \\ t=3.742s & & t=4.112s \end{array}$$

Therefore, the first instant is t = 0.3425 s.

b) The formula for the velocity is

$$\begin{aligned} v &= A\omega\cos(\omega t+\phi) \\ &= 0.2m\cdot 5\tfrac{rad}{s}\cdot\cos\left(5\tfrac{rad}{s}\cdot t+\tfrac{\pi}{4}rad\right) \\ &= 1\tfrac{m}{s}\cdot\cos\left(5\tfrac{rad}{s}t+\tfrac{\pi}{4}rad\right) \end{aligned}$$

The instant the velocity is -0.6 m/s is found with

$$-0.6\tfrac{m}{s} = 1\tfrac{m}{s}\cdot\cos\left(5\tfrac{rad}{s}\cdot t+\tfrac{\pi}{4}rad\right)$$

$$-0.6 = \cos\left(5\tfrac{rad}{s}\cdot t+\tfrac{\pi}{4}rad\right)$$

$$\begin{array}{rcl} 2.2143 = 5s^{-1}\cdot t+\frac{\pi}{4} & and & -2.2143 = 5s^{-1}\cdot t+\frac{\pi}{4} \\ 1.4289 = 5s^{-1}\cdot t & and & -2.9997 = 5s^{-1}\cdot t \\ t = 0.2858s & and & t=-0.5999s \end{array}$$

The period can then be added (or removed) to each of these answers to find all the instants where the velocity is v = -0.6 m/s. We thus obtain

$$\begin{array}{ccc} t=0.2858s & & t=-0.5999s \\ & \downarrow +1.2566s & \\ t=1.542s & & t=0.6567s \\ & \downarrow +1.2566s & \\ t=2.799s & & t=1.913s \\ & \downarrow +1.2566s & \\ t=4.056s & & t=3.170s \end{array}$$

Therefore, the first instant is t = 0.2858 s.

11. When the object is at x = 8 cm, we have

$$0.08m = 0.16m \cdot \sin\left(10\tfrac{rad}{s} \cdot t + \tfrac{\pi}{2} rad\right)$$

$$\frac{1}{2} = \sin\left(10\tfrac{rad}{s} \cdot t + \tfrac{\pi}{2} rad\right)$$

$$\frac{\pi}{6} = 10s^{-1} \cdot t + \tfrac{\pi}{2} \qquad and \qquad \frac{5\pi}{6} = 10s^{-1} \cdot t + \tfrac{\pi}{2}$$

$$-\frac{\pi}{3} = 10s^{-1} \cdot t \qquad and \qquad \frac{\pi}{3} = 10s^{-1} \cdot t$$

$$t = -\frac{\pi}{30} s \qquad and \qquad t = \frac{\pi}{30} s$$

Only one of these two answers is good since it is said that the velocity must be positive. The formula for the velocity as a function of time is

$$\begin{aligned} v &= A\omega\cos\left(\omega t + \phi\right) \\ &= 0.16m \cdot 10s^{-1} \cdot \cos\left(10s^{-1} \cdot t + \tfrac{\pi}{2} rad\right) \\ &= 1.6\tfrac{m}{s} \cdot \cos\left(10s^{-1} \cdot t + \tfrac{\pi}{2} rad\right) \end{aligned}$$

At t = -π/30 s, the velocity is

$$\begin{aligned} v &= 1.6\tfrac{m}{s} \cdot \cos\left(10\tfrac{rad}{s} \cdot \left(-\tfrac{\pi}{30} s\right) + \tfrac{\pi}{2} rad\right) \\ &= 1.6\tfrac{m}{s} \cdot \cos\left(\tfrac{\pi}{6} rad\right) \\ &= 1.3856\tfrac{m}{s} \end{aligned}$$

whereas at t = π/30 s, the velocity is

$$\begin{aligned} v &= 1.6\tfrac{m}{s} \cdot \cos\left(10\tfrac{rad}{s} \cdot \tfrac{\pi}{30} s + \tfrac{\pi}{2} rad\right) \\ &= 1.6\tfrac{m}{s} \cdot \cos\left(\tfrac{5\pi}{6} rad\right) \\ &= -1.3856\tfrac{m}{s} \end{aligned}$$

As a positive velocity is required, t = -π/30 s = -0.1047 s is the right answer.

The period can then be added (or removed) to this answer to find all the instants where the object is at x = 8 cm with a positive velocity. The period is

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ &= \frac{2\pi}{10s^{-1}} \\ &= 0.6283s \end{aligned}$$

The instants are thus

-0.1047 s, 0.5236 s, 1.1519 s, 1.7802 s,…

The first positive instant is thus 0.5236 s.

12. In the formula

$$x = A\sin\left(\omega t + \phi\right)$$

we need to find A, ω and ϕ.

ω is

$$\begin{aligned} \omega &= \frac{2\pi}{T} \\ &= \frac{2\pi}{0.5s} \\ &= 4\pi \tfrac{rad}{s} \end{aligned}$$

The amplitude can be found with the maximum acceleration

$$\begin{aligned} a_{\max} &= \omega^2 A \\ 32\tfrac{m}{s^2} &= \left(4\pi s^{-1}\right)^2 A \\ A &= 0,2026m \end{aligned}$$

Finally, the phase constant can be found. As the speed is 0 when the acceleration is at its maximal value, we have

$$\tan(\omega t+\phi)=\frac{x\omega}{v}$$
$$\tan(0+\phi)=\frac{0.2026m\cdot 4\pi s^{-1}}{0\frac{m}{s}}$$
$$\tan(\phi)=\infty$$
$$\phi=\tfrac{\pi}{2}rad$$

(The value of x is positive at t = 0 because the acceleration is negative. a and x always have opposite signs according to $a=-\omega^2 x$.)

Therefore, the equation of motion is

$$x=0.2026m\cdot\sin\left(4\pi\tfrac{rad}{s}\cdot t+\tfrac{\pi}{2}rad\right)$$

13. As the motion is symmetrical with respect to x = 0, the time to go from x = -10 cm to x = 0 is the same as the time to go from x = 0 to x = 10 cm. So, we're going to find the time to go from x = 0 to x = 10 cm and multiply by 2 at the end.

It remains to find when the object is at x = 0 and when it is at x = 10 cm.

Since it is not specified when is t = 0 is this problem, we can choose it. So, we're going to choose that at t = 0, the object is at x = 0 and has a positive speed (so that it's heading towards x =10 cm). Well, we have found that the object is at x = 0 when t = 0.

It only remains to find when the object is at x = 10 cm. To do this, we need the equation of motion

$$x=A\sin(\omega t+\phi)$$

We know that A = 20 cm but ω are ϕ must be calculated.

Since the period is 6 s, ω is

$$\omega=\frac{2\pi}{T}$$
$$=\frac{2\pi}{6s}$$
$$=\frac{\pi}{3}\tfrac{rad}{s}$$

Since the object is at x = 0 and has a positive speed at t = 0, the phase constant is

$$\tan(\omega t+\phi)=\frac{x\omega}{v}$$
$$\tan(0+\phi)=\frac{0m\cdot\frac{\pi}{3}s^{-1}}{v}$$
$$\tan(\phi)=0$$
$$\phi=0rad$$

The equation of motion is

$$x=0.2m\cdot\sin\left(\tfrac{\pi}{3}\tfrac{rad}{s}\cdot t\right)$$

Now, let's find when the object is at x = 10 cm. We then have

$$0.1m=0.2m\cdot\sin\left(\tfrac{\pi}{3}\tfrac{rad}{s}\cdot t\right)$$
$$\frac{1}{2}=\sin\left(\tfrac{\pi}{3}\tfrac{rad}{s}\cdot t\right)$$
$$\frac{\pi}{6}=\frac{\pi}{3}s^{-1}\cdot t \qquad and \qquad \frac{5\pi}{6}=\frac{\pi}{3}s^{-1}\cdot t$$
$$t=\frac{1}{2}s \qquad and \qquad t=\frac{5}{2}s$$

It then takes 1/2 s for the object to move from x = 0 cm to x = 10 cm. (The other answer corresponds to the object starting from x = 0 cm, going to x = 20 cm and then returning to x = 10 cm.)

It then takes 1/2 s to travel from x = – 10 cm to x = 0 cm, and then another 1/2 s to travel from x = 0 cm to x = 10 cm. The total time is, therefore, 1 s.

14. a) The amplitude will be found with

$$A^2=x^2+\left(\frac{v}{\omega}\right)^2$$

We need the value of ω. The value of ω is

$$\omega = \sqrt{\frac{k}{m}}$$
$$= \sqrt{\frac{81\frac{N}{m}}{0.25kg}}$$
$$= 18\tfrac{rad}{s}$$

Thus, the amplitude is

$$A^2 = x^2 + \left(\frac{v}{\omega}\right)^2$$
$$= (0.1m)^2 + \left(\frac{-2\frac{m}{s}}{18s^{-1}}\right)^2$$
$$= 0.022346m^2$$
$$A = 0.1495m$$

b) The phase constant is

$$\tan(\omega t + \phi) = \frac{x\omega}{v}$$
$$\tan\left(18\tfrac{rad}{s}\cdot 1s + \phi\right) = \frac{0.1m \cdot 18s^{-1}}{-2\frac{m}{s}}$$
$$\tan(18rad + \phi) = -0.9$$
$$18 + \phi = 2.4088rad$$

Note that the answer given by the calculator is -0.7328 rad, but π must be added since the value of v is negative (or this is a matter of quadrants, depending on your way of looking at this issue).

We then have

$$18 + \phi = 2.4088rad$$
$$\phi = -15.591rad$$

As many 2π as we want can be added or subtracted from this value if needed.

c) The equation of motion is

$$x = 0.1495m \cdot \sin\left(18\tfrac{rad}{s} \cdot t - 15.591rad\right)$$

15. We have

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$
$$10Hz = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

When the mass is changed, we have

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$
$$6Hz = \frac{1}{2\pi}\sqrt{\frac{k}{m+0.1kg}}$$

We have 2 equations and 2 unknowns.

a) *m* is found with the ratio of these 2 equations.

$$\frac{10Hz}{6hz} = \frac{\frac{1}{2\pi}\sqrt{\frac{k}{m}}}{\frac{1}{2\pi}\sqrt{\frac{k}{m+0.1kg}}}$$
$$\frac{5}{3} = \sqrt{\frac{m+0.1kg}{m}}$$
$$\frac{25}{9} = \frac{m+0.1kg}{m}$$
$$\tfrac{25}{9}m = m + 0.1kg$$
$$\tfrac{16}{9}m = 0.1kg$$
$$m = 0.05625kg$$

b) The spring constant is thus

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$
$$10Hz = \frac{1}{2\pi}\cdot\sqrt{\frac{k}{0.05625kg}}$$
$$k = 222.1\tfrac{N}{m}$$

16. The period will be found with

$$T = 2\pi\sqrt{\frac{m}{k}}$$

The mass is known but not the spring constant. It can be found with

$$\begin{aligned} mg &= ky_0 \\ 0.2kg \cdot 9.8\tfrac{N}{kg} &= k \cdot 0.1m \\ k &= 19.6\tfrac{N}{m} \end{aligned}$$

Therefore, the oscillation period is

$$\begin{aligned} T &= 2\pi\sqrt{\frac{m}{k}} \\ &= 2\pi \cdot \sqrt{\frac{0.2kg}{19.6\frac{N}{m}}} \\ &= 0.6347s \end{aligned}$$

17. a)

Passing from one side to the other of the motion, the amplitude is travelled twice (from x = -A to x = 0, and then from x = 0 to x = A). Therefore, the amplitude is 6 cm.

b) This movement is only on the half of a full oscillation. Therefore, the 0.8s is half the period and the period of 1.6 sec.

c) The maximum speed is

$$v_{\max} = A\omega$$

ω is found with the period.

$$\begin{aligned}\omega &= \frac{2\pi}{T} \\ &= \frac{2\pi}{1.6s} \\ &= 3.927 \tfrac{rad}{s}\end{aligned}$$

The maximum speed is, therefore,

$$\begin{aligned}v_{max} &= A\omega \\ &= 0.06m \cdot 3.927s^{-1} \\ &= 0.2356 \tfrac{m}{s}\end{aligned}$$

d) The maximum acceleration is

$$\begin{aligned}a_{max} &= A\omega^2 \\ &= 0.06m \cdot \left(3.927s^{-1}\right)^2 \\ &= 0.9253 \tfrac{m}{s^2}\end{aligned}$$

18. The mass will be found with

$$\omega = \sqrt{\frac{k}{m}}$$

k is known but not ω. ω can be found with

$$\begin{aligned}v_{max} &= A\omega \\ 4 \tfrac{m}{s} &= 0.2m \cdot \omega \\ \omega &= 20 \tfrac{rad}{s}\end{aligned}$$

Therefore,

$$\omega = \sqrt{\frac{k}{m}}$$
$$20s^{-1} = \sqrt{\frac{250\frac{N}{m}}{m}}$$
$$400s^{-2} = \frac{250\frac{N}{m}}{m}$$
$$m = 0,625kg$$

19. The period will be found with

$$T = 2\pi\sqrt{\frac{m}{k}}$$

The mass is known, but not *k*.

Since *F* = -*kx,* the slope of the graph is equal to -*k*. As the slope is

$$slope = \frac{-8N}{0,4m}$$
$$= -20\frac{N}{m}$$

Thus, *k* is 20 N/m. Therefore, the period is

$$T = 2\pi\sqrt{\frac{m}{k}}$$
$$= 2\pi \cdot \sqrt{\frac{0.2kg}{20\frac{N}{m}}}$$
$$= 0.6283s$$

20. The phase constant is found with

$$\tan\left(\omega t + \phi\right) = \frac{\omega x}{v}$$

To calculate it, ω and v are needed. ω can be found with the period.

$$T = \frac{2\pi}{\omega}$$
$$2s = \frac{2\pi}{\omega}$$
$$\omega = \pi \tfrac{rad}{s}$$

The velocity at t = 0 can be found with the position at t = 0.

$$x^2 + \left(\frac{v}{\omega}\right)^2 = A^2$$
$$\left(-0.08m\right)^2 + \left(\frac{v}{\pi s^{-1}}\right)^2 = \left(0.12m\right)^2$$
$$v = 0.28099 \tfrac{m}{s}$$

Since the velocity is positive, v = 0.28099 m/s.

Therefore, the phase constant is

$$\tan\left(\omega t + \phi\right) = \frac{\omega x}{v}$$
$$\tan\left(0 + \phi\right) = \frac{\pi \tfrac{rad}{s} \cdot \left(-0.08m\right)}{0.28099 \tfrac{m}{s}}$$
$$\tan\phi = -0.8944$$
$$\phi = -0.7297$$

21. Without any water, the spring is stretched a bit to support the bucket. Then

$$mg = ky_0$$
$$2kg \cdot 9.8 \tfrac{N}{kg} = ky_0$$
$$ky_0 = 19.6N$$

When water is added, the mass increases and the elongation of the spring increases by 12 cm. Then

$$\left(2kg + m_{water}\right) g = k\left(y_0 + 0.12m\right)$$
$$2kg \cdot 9.8 \tfrac{N}{kg} + m_{water} \cdot 9.8 \tfrac{N}{kg} = ky_0 + k \cdot 0.12m$$
$$19.6N + m_{water} \cdot 9.8 \tfrac{N}{kg} = ky_0 + k \cdot 0.12m$$

But since ky_0 = 19.6 N, the equation becomes

$$19.6N + m_{water} \cdot 9.8\tfrac{N}{kg} = ky_0 + k \cdot 0{,}12m$$
$$19.6N + m_{water} \cdot 9.8\tfrac{N}{kg} = 19.6N + k \cdot 0.12m$$
$$m_{water} \cdot 9.8\tfrac{N}{kg} = k \cdot 0.12m$$

It is also known that the period of oscillation is 2.4 s. Therefore,

$$T = 2\pi\sqrt{\frac{m}{k}}$$
$$2.4s = 2\pi\sqrt{\frac{2kg + m_{water}}{k}}$$

If we solve for k in this equation

$$\frac{2.4s}{2\pi} = \sqrt{\frac{2kg + m_{eau}}{k}}$$
$$0.382s = \sqrt{\frac{2kg + m_{water}}{k}}$$
$$0.1459s^2 = \frac{2kg + m_{water}}{k}$$
$$k = \frac{2kg + m_{water}}{0.1459s^2}$$

and then substitute this value into the other equation

$$m_{water} \cdot 9.8\tfrac{N}{kg} = k \cdot 0.12m$$

we obtain

$$m_{eau} \cdot 9.8\tfrac{N}{kg} = \frac{2kg + m_{eau}}{0.1459s^2} \cdot 0.12m$$

Then, we solve for m_{water} to obtain the answer.

$$m_{water} \cdot 1.4298m = \left(2kg + m_{water}\right) \cdot 0.12m$$
$$m_{water} \cdot 1.4298m = 2kg \cdot 0{,}12m + m_{water} \cdot 0.12m$$
$$m_{water} \cdot 1.3098m = 2kg \cdot 0.12m$$
$$m_{water} = 0.1832kg$$

22. a) Since half of a cycle lasts 0.6 seconds, the period is 1.2 seconds. Thus,

$$T = 2\pi\sqrt{\frac{m}{k}}$$
$$1.2s = 2\pi \cdot \sqrt{\frac{2kg}{k}}$$
$$k = 54.83\tfrac{N}{m}$$

b) Let's find out by how much the spring has to stretch to reach the equilibrium position

$$mg = ky_0$$
$$y_0 = \frac{mg}{k}$$
$$y_0 = \frac{2kg \cdot 9.8\frac{N}{kg}}{54.83\frac{N}{m}}$$
$$y_0 = 0.3575m$$

But the distance between the highest position and the equilibrium position is equal to the amplitude. Therefore, the amplitude is 35.75 cm.

c) The maximum speed is

$$v_{\max} = A\omega$$

Here

$$\omega = \frac{2\pi}{T}$$
$$= \frac{2\pi}{1,2s}$$
$$= \frac{5\pi}{3}\frac{rad}{s}$$

Thus

$$\begin{aligned} v_{max} &= A\omega \\ &= 0.3575m \cdot \frac{5\pi}{3} s^{-1} \\ &= 1.872 \tfrac{m}{s} \end{aligned}$$

d) The velocity is given by

$$\begin{aligned} v &= A\omega \cos(\omega t + \phi) \\ &= v_{max} \cos(\omega t + \phi) \end{aligned}$$

First, let's find the phase constant. We know that the speed is zero initially. Therefore

$$\tan(\omega t + \phi) = \frac{x\omega}{v}$$

$$\tan \phi = \infty$$

$$\phi = \frac{\pi}{2} rad$$

The velocity formula is thus

$$\begin{aligned} v &= A\omega \cos(\omega t + \phi) \\ &= 1,872 \tfrac{m}{s} \cdot \cos\left(\tfrac{5\pi}{3} \tfrac{rad}{s} \cdot t + \tfrac{\pi}{2} rad\right) \end{aligned}$$

If the velocity is 1 m/s (positive since it is directed upwards), then

$$1 \tfrac{m}{s} = 1.872 \tfrac{m}{s} \cdot \cos\left(\tfrac{5\pi}{3} \tfrac{rad}{s} \cdot t + \tfrac{\pi}{2} rad\right)$$

$$0.5349 = \cos\left(\tfrac{5\pi}{3} \tfrac{rad}{s} \cdot t + \tfrac{\pi}{2} rad\right)$$

$$\pm 1.007 = \tfrac{5\pi}{3} s^{-1} \cdot t + \tfrac{\pi}{2}$$

With the positive value, the result is

$$1,007 = \tfrac{5\pi}{3} s^{-1} \cdot t + \tfrac{\pi}{2}$$

$$-0,5637 = \tfrac{5\pi}{3} s^{-1} \cdot t$$

$$t = -0,1077 s$$

Adding the period, this time is *t* = 1.0923 s.

With the negative value, the result is

$$-1,007 = \tfrac{5\pi}{3} s^{-1} \cdot t + \tfrac{\pi}{2}$$
$$-2,578 = \tfrac{5\pi}{3} s^{-1} \cdot t$$
$$t = -0,4923s$$

Adding the period, this time is t = 0.7077 s.

Thus, it can be seen that it will happen for the first time at t = 0.7077 s.

23. a) The energy is

$$\begin{aligned} E_{mec} &= \frac{1}{2} kA^2 \\ &= \frac{1}{2} \cdot 250 \tfrac{N}{m} \cdot (0.2m)^2 \\ &= 5J \end{aligned}$$

b) At t = 5 s, the position is

$$\begin{aligned} x &= 0.2m \cdot \sin\left(5 \tfrac{rad}{s} \cdot 5s + \tfrac{\pi}{4} rad\right) \\ &= 0.12146m \end{aligned}$$

The spring energy is thus

$$\begin{aligned} U_{Sp} &= \frac{1}{2} kx^2 \\ &= \frac{1}{2} \cdot 250 \tfrac{N}{m} \cdot (0.12146m)^2 \\ &= 1.844J \end{aligned}$$

c) The kinetic energy can be found with

$$E_{mec} = E_k + U_R$$
$$5J = E_k + 1.844J$$
$$E_k = 3.156J$$

24. a) We have

$$E_{mec} = \frac{1}{2}kA^2$$
$$30J = \frac{1}{2}\cdot k\cdot(0.8m)^2$$
$$k = 93.75\tfrac{N}{m}$$

b) With a 3 s period, the mass is

$$T = 2\pi\sqrt{\frac{m}{k}}$$
$$3s = 2\pi\cdot\sqrt{\frac{m}{93.75\frac{N}{m}}}$$
$$m = 21.37kg$$

c) ω is

$$\begin{aligned}\omega &= \frac{2\pi}{T}\\ &= \frac{2\pi}{3s}\\ &= 2.094\tfrac{rad}{s}\end{aligned}$$

The maximum speed is thus

$$\begin{aligned}v_{\max} &= A\omega\\ &= 0.8m\cdot 2.094s^{-1}\\ &= 1.6755\tfrac{m}{s}\end{aligned}$$

25. a) The kinetic energy will be found with

$$E_k + U = E_{mec}$$

The mechanical energy is 5 J. We thus need the spring energy. The spring energy is

$$U = \frac{1}{2}kx^2$$

We know the position but not the spring constant. This spring constant can be found with the amplitude.

$$E_{mec} = \frac{1}{2}kA^2$$
$$5J = \frac{1}{2}\cdot k\cdot(0.1m)^2$$
$$k = 1000\tfrac{N}{m}$$

Thus, the spring energy is

$$\begin{aligned} U &= \frac{1}{2}kx^2 \\ &= \frac{1}{2}\cdot 1000\tfrac{N}{m}\cdot(0.06m)^2 \\ &= 1.8J \end{aligned}$$

With the conservation of energy, we then have

$$E_k + U = E_{mec}$$
$$E_k + 1.8J = 5J$$
$$E_k = 3.2J$$

b) The energy is

$$\begin{aligned} U_{sp} &= \frac{1}{2}kx^2 \\ &= \frac{1}{2}\cdot 1000\tfrac{N}{m}\cdot(0.04m)^2 \\ &= 0.8J \end{aligned}$$

26. We have

$$E_{mec} = E_k + U$$

If $E_k = U$, then

$$E_{mec} = U + U$$
$$E_{mec} = 2U$$
$$\frac{1}{2}m\omega^2A^2 = 2\cdot\frac{1}{2}m\omega^2x^2$$
$$A^2 = 2\cdot x^2$$

This gives

$$x = \pm \frac{A}{\sqrt{2}} = \pm \frac{0,25m}{\sqrt{2}}$$

Now, we will find out when this happens. Let's start with the positive value of *x*.

$$\begin{aligned} x &= 0.25m \cdot \sin\left(10\tfrac{rad}{s} \cdot t + \pi rad\right) \\ \frac{0.25m}{\sqrt{2}} &= 0.25m \cdot \sin\left(10\tfrac{rad}{s} \cdot t + \pi rad\right) \\ \frac{1}{\sqrt{2}} &= \sin\left(10\tfrac{rad}{s} \cdot t + \pi rad\right) \end{aligned}$$

$$\begin{array}{ccc} \frac{\pi}{4} = 10s^{-1} \cdot t + \pi & \text{and} & \frac{3\pi}{4} = 10s^{-1} \cdot t + \pi \\ \frac{-3\pi}{4} = 10s^{-1} \cdot t & \text{and} & \frac{-\pi}{4} = 10s^{-1} \cdot t \\ t = -0.2356s & \text{and} & t = -0.0785s \end{array}$$

The period is then added to these two answers to obtain positive values of *t*. As the period is

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ &= \frac{2\pi}{10s^{-1}} \\ &= 0.6283s \end{aligned}$$

the times are

$$\begin{array}{ccc} t = -0.2356s & & t = -0.0785s \\ & \downarrow +0.6283s & \\ t = 0.3927s & & t = 0.5498s \end{array}$$

Now, let's have a look at the negative value of *x*.

$$x = 0.25m \cdot \sin\left(10\tfrac{rad}{s} \cdot t + \pi rad\right)$$
$$-\frac{0.25m}{\sqrt{2}} = 0.25m \cdot \sin\left(10\tfrac{rad}{s} \cdot t + \pi rad\right)$$
$$-\frac{1}{\sqrt{2}} = \sin\left(10\tfrac{rad}{s} \cdot t + \pi rad\right)$$
$$-\frac{\pi}{4} = 10s^{-1} \cdot t + \pi \qquad and \qquad \frac{5\pi}{4} = 10s^{-1} \cdot t + \pi$$
$$\frac{-5\pi}{4} = 10s^{-1} \cdot t \qquad and \qquad \frac{\pi}{4} = 10s^{-1} \cdot t$$
$$t = -0.3927s \qquad and \qquad t = 0.0785s$$

The period is then added to the first answer to obtain a positive value. The result is 0.2356 s.

Therefore, the instants are, in ascending order: 0.0785s, 0.2356, 0.3927, 0.5498 s.

The first instant is t = 0.0785 s.

27. a) We start with

$$A^2 = x^2 + \left(\frac{v}{\omega}\right)^2$$

Since the speed is equal to one quarter of the maximum speed, we have

$$v = \frac{v_{\max}}{4}$$
$$v = \frac{A\omega}{4}$$

Therefore,

$$A^2 = x^2 + \left(\frac{A\omega/4}{\omega}\right)^2$$
$$A^2 = x^2 + \left(\frac{A}{4}\right)^2$$
$$A^2 = x^2 + \frac{A^2}{16}$$
$$\frac{15A^2}{16} = x^2$$
$$x = \pm\sqrt{\frac{15}{16}}A$$
$$x = \pm\sqrt{\frac{15}{16}} \cdot 0.12m$$
$$x = \pm 0.1162m$$

b) We start with

$$E_{mec} = E_k + U_R$$

Since $E_k = ½U$, we have

$$E_{mec} = \frac{1}{2}U_R + U_R$$
$$= \frac{3}{2}U_R$$

This gives

$$\frac{1}{2}kA^2 = \frac{3}{2}\left(\frac{1}{2}kx^2\right)$$
$$A^2 = \frac{3}{2}x^2$$
$$x = \pm\sqrt{\frac{2}{3}}A$$
$$x = \pm\sqrt{\frac{2}{3}} \cdot 0.12m$$
$$x = \pm 0.09798m$$

28. a) The period is

$$\begin{aligned} T &= 2\pi\sqrt{\frac{m}{k}} \\ &= 2\pi \cdot \sqrt{\frac{2kg}{200\frac{N}{m}}} \\ &= 0.6283s \end{aligned}$$

b) The amplitude will be found with

$$A^2 = y^2 + \left(\frac{v}{\omega}\right)^2$$

This y is the position of the mass from the equilibrium position. The stretching of the spring at the equilibrium position is

$$\begin{aligned} y_0 &= \frac{mg}{k} \\ &= \frac{2kg \cdot 9.8\frac{N}{kg}}{200\frac{N}{m}} \\ &= 0.098m \end{aligned}$$

Since the spring is not initially stretched, the mass is at y = 9.8 cm from the equilibrium position. With a zero initial velocity, the amplitude is

$$\begin{aligned} A^2 &= y^2 + \left(\frac{v}{\omega}\right)^2 \\ A^2 &= (0.098m)^2 + \left(\frac{0\frac{m}{s}}{\omega}\right)^2 \\ A &= 0.098m \end{aligned}$$

Alternate Solution

The initial position of the block corresponds to the highest point of the motion since its speed is zero. The block will then descend until the lowest point is reached where the speed will become zero again. This motion corresponds to 2 times the amplitude, because the block will travel once the amplitude from the highest point to the point of equilibrium, and once again from the equilibrium position to the lowest point.

The total displacement of the block down to the lowest point can be found with energy conservation. Since the system is composed of a block and a spring, the mechanical energy is

$$E = \frac{1}{2}mv^2 + mgy + \frac{1}{2}kx^2$$

Initially (block at rest at its highest point), the energy is, by placing the $y = 0$ at the initial position of the block,

$$\begin{aligned} E &= \frac{1}{2}\cdot 2kg \cdot \left(0\tfrac{m}{s}\right)^2 + 2kg \cdot 9.8\tfrac{N}{kg}\cdot 0m + \frac{1}{2}\cdot 200\tfrac{N}{m}\cdot (0m)^2 \\ &= 0 \end{aligned}$$

Then, the block travels downwards a distance *d*, which stretches the spring by a length *d*. The energy is then

$$\begin{aligned} E' &= \frac{1}{2}\cdot 2kg \cdot \left(0\tfrac{m}{s}\right)^2 + 2kg \cdot 9.8\tfrac{N}{kg}\cdot (-d) + \frac{1}{2}\cdot 200\tfrac{N}{m}\cdot d^2 \\ &= -19.6N\cdot d + 100\tfrac{N}{m}\cdot d^2 \end{aligned}$$

According to the law of mechanical energy conservation, we then have

$$\begin{aligned} E &= E' \\ 0J &= -19.6N\cdot d + 100\tfrac{N}{m}\cdot d^2 \\ 19.6N\cdot d &= 100\tfrac{N}{m}\cdot d^2 \\ 19.6N &= 100\tfrac{N}{m}\cdot d \\ d &= 0.196m \end{aligned}$$

The amplitude is half this distance, which means that the amplitude is 9.8 cm.

29. As the pendulum is vertical 2 times per cycle, the period of the pendulum must be 4 s. Therefore

$$\begin{aligned} T &= 2\pi\sqrt{\frac{l}{g}} \\ 4s &= 2\pi\cdot\sqrt{\frac{l}{9.8\frac{N}{kg}}} \\ l &= 3.972m \end{aligned}$$

30. If the period is 2 s on Earth, then the length of the string is

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$2s = 2\pi \cdot \sqrt{\frac{l}{9.8\frac{N}{kg}}}$$

$$l = 0.9929m$$

The period on the Moon would, therefore, be

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$T = 2\pi \cdot \sqrt{\frac{0.9929m}{1.6\frac{N}{kg}}}$$

$$T = 4.95s$$

31. a) The period is

$$\begin{aligned} T &= 2\pi\sqrt{\frac{l}{g}} \\ &= 2\pi \cdot \sqrt{\frac{2m}{9.8\frac{N}{kg}}} \\ &= 2.838s \end{aligned}$$

b) The maximum speed is

$$v_{\max} = A\omega$$

The amplitude is needed. This amplitude is

$$\begin{aligned} A &= \theta_{\max} l \\ &= \left(10°\frac{2\pi rad}{360°}\right) \cdot 2m \\ &= 0.3491m \end{aligned}$$

The angular frequency is also needed. It is

$$\begin{aligned}\omega &= \sqrt{\frac{g}{l}} \\ &= \sqrt{\frac{9.8\frac{N}{kg}}{2m}} \\ &= 2.214\frac{rad}{s}\end{aligned}$$

Therefore, the maximum speed is

$$\begin{aligned}v_{\max} &= 0.3491m \cdot 2.214s^{-1} \\ &= 0.7727\frac{m}{s}\end{aligned}$$

32. The velocity is found with

$$A^2 = x^2 + \left(\frac{v}{\omega}\right)^2$$

ω is

$$\begin{aligned}\omega &= \frac{2\pi}{T} \\ &= \frac{2\pi}{1,6s} \\ &= 3,927\frac{rad}{s}\end{aligned}$$

To find the amplitude, the length of the rope is needed. This length with is found with

$$\begin{aligned}T &= 2\pi\sqrt{\frac{l}{g}} \\ 1.6s &= 2\pi \cdot \sqrt{\frac{l}{9.8\frac{N}{kg}}} \\ l &= 0.6355m\end{aligned}$$

Therefore, the amplitude is

$$\begin{aligned} A &= \theta_{\max} l \\ &= \left(12° \frac{2\pi rad}{360°}\right) \cdot 0.6355m \\ &= 0.1331m \end{aligned}$$

The position at $\theta = 8°$ is

$$\begin{aligned} x &= \theta l \\ &= \left(8° \frac{2\pi rad}{360°}\right) \cdot 0.6355m \\ &= 0.0887m \end{aligned}$$

Therefore,

$$\begin{aligned} A^2 &= x^2 + \left(\frac{v}{\omega}\right)^2 \\ (0.1311m)^2 &= (0.0887m)^2 + \left(\frac{v}{3.927s^{-1}}\right)^2 \\ v &= 0.3896 \tfrac{m}{s} \end{aligned}$$

33. In the equation

$$\theta = \theta_{\max} \sin(\omega t + \phi)$$

We need to calculate the amplitude, ω and ϕ.

ω is

$$\begin{aligned} \omega &= \sqrt{\frac{g}{l}} \\ &= \sqrt{\frac{9.8 \frac{N}{kg}}{1.2m}} \\ &= 2.8577 \tfrac{rad}{s} \end{aligned}$$

For the angular amplitude, we'll start with *A*. The amplitude is found with

$$A^2 = x^2 + \left(\frac{v}{\omega}\right)^2$$

We know that the velocity is -20 cm/s. The position is

$$\begin{aligned} x &= \theta l \\ &= \left(6° \frac{2\pi rad}{360°}\right) \cdot 1.2m \\ &= 0.1257m \end{aligned}$$

Therefore,

$$\begin{aligned} A^2 &= x^2 + \left(\frac{v}{\omega}\right)^2 \\ &= (0.1257m)^2 + \left(\frac{0.2\frac{m}{s}}{2.8577s^{-1}}\right)^2 \\ &= 0.02069m^2 \\ A &= 0.1438m \end{aligned}$$

The angular amplitude is then

$$\begin{aligned} A &= \theta_{max} l \\ 0.1438m &= \theta_{max} \cdot 1.2m \\ \theta_{max} &= 0.1199rad \\ \theta_{max} &= 6.868° \end{aligned}$$

The phase constant is

$$\begin{aligned} \tan(\omega t + \phi) &= \frac{x\omega}{v} \\ \tan(0 + \phi) &= \frac{0.1257m \cdot 2.8577s^{-1}}{-0.2\frac{m}{s}} \\ \phi &= 2.079rad \end{aligned}$$

(π was added since the velocity is negative.)

The equation is thus

$$\theta = 6.868° \cdot \sin\left(2.858\tfrac{rad}{s} \cdot t + 2.079rad\right)$$

34. a) We know that the maximum speed is 0.6. Since $v_{max} = A\omega$, we have

$$A\omega = 0.6\tfrac{m}{s}$$

Using the formula for the angular amplitude

$$A = \theta_{\max} l$$

and of ω

$$\omega = \sqrt{\frac{g}{l}}$$

We arrive at

$$\begin{aligned}
(\theta_{\max} l)\sqrt{\frac{g}{l}} &= 0.6\tfrac{m}{s}\\
\theta_{\max}\frac{l}{\sqrt{l}}\sqrt{g} &= 0.6\tfrac{m}{s}\\
\theta_{\max}\sqrt{l}\sqrt{g} &= 0.6\tfrac{m}{s}\\
\left(15°\tfrac{2\pi rad}{360°}\right)\cdot\sqrt{l}\cdot\sqrt{9.8\tfrac{N}{kg}} &= 0.6\tfrac{m}{s}\\
l &= 0.536m
\end{aligned}$$

b) The period is

$$\begin{aligned}
T &= 2\pi\sqrt{\frac{l}{g}}\\
&= 2\pi\cdot\sqrt{\frac{0.536m}{9.8\tfrac{N}{kg}}}\\
&= 1.47s
\end{aligned}$$

35. The period will be found with

$$T = \frac{2\pi}{\omega}$$

ω is

$$\omega = \sqrt{\frac{mgd}{I}}$$

d is the distance between the centre of mass and the axis of rotation. This distance is equal to the radius of the disc, thus to 20 cm.

The moment of inertia is

$$\begin{aligned} I &= \frac{3}{2} mR^2 \\ &= \frac{3}{2} \cdot 5kg \cdot (0.2m)^2 \\ &= 0.3kgm^2 \end{aligned}$$

Therefore,

$$\begin{aligned} \omega &= \sqrt{\frac{mgd}{I}} \\ &= \sqrt{\frac{5kg \cdot 9.8 \frac{N}{kg} \cdot 0.2m}{0.3kgm^2}} \\ &= 5.715 \frac{rad}{s} \end{aligned}$$

The period is then

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ &= \frac{2\pi}{5.715s^{-1}} \\ &= 1.099s \end{aligned}$$

36. The period will be found with

$$T = \frac{2\pi}{\omega}$$

ω is

$$\omega = \sqrt{\frac{mgd}{I}}$$

d is the distance between the centre of mass and the axis of rotation. This distance is equal to 75 cm.

The moment of inertia is

$$\begin{aligned} I &= \frac{1}{12} mL^2 + mh^2 \\ &= \frac{1}{12} \cdot 2kg \cdot (2m)^2 + 2kg \cdot (0.75m)^2 \\ &= 1.792 kgm^2 \end{aligned}$$

Therefore,

$$\begin{aligned} \omega &= \sqrt{\frac{mgd}{I}} \\ &= \sqrt{\frac{2kg \cdot 9.8 \frac{N}{kg} \cdot 0.75m}{1.792 kgm^2}} \\ &= 2.864 \frac{rad}{s} \end{aligned}$$

The period is then

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ &= \frac{2\pi}{2.864 s^{-1}} \\ &= 2.194 s \end{aligned}$$

37. The moment of inertia will be found with

$$\omega = \sqrt{\frac{mgd}{I}}$$

ω is

$$\begin{aligned} \omega &= \frac{2\pi}{T} \\ &= \frac{2\pi}{1.335 s} \\ &= 4.707 \frac{rad}{s} \end{aligned}$$

Therefore,

$$\omega = \sqrt{\frac{mgd}{I}}$$

$$4.707\,\tfrac{rad}{s} = \sqrt{\frac{0.9kg \cdot 9.8\tfrac{N}{kg} \cdot 0.43m}{I}}$$

$$I = 0.1712kgm^2$$

38. The oscillation period is

$$T = 2\pi\sqrt{\frac{I}{mgd}}$$

In this case, the moment of inertia is

$$I = \frac{1}{12}mL^2 + md^2$$

and the period becomes

$$T = 2\pi\sqrt{\frac{\frac{1}{12}mL^2 + md^2}{mgd}}$$

$$= \frac{2\pi}{\sqrt{g}}\sqrt{\frac{\frac{1}{12}L^2 + d^2}{d}}$$

$$= \frac{2\pi}{\sqrt{g}}\sqrt{\frac{L^2}{12d} + d}$$

To obtain the minimum period, the term in the square root must be as small as possible. We, therefore, have to find the value of d that gives the smallest value to

$$\frac{L^2}{12d} + d$$

There is a minimum when the derivative of this function is zero. Therefore,

$$-\frac{L^2}{12d^2}+1=0$$
$$\frac{L^2}{12d^2}=1$$
$$L^2=12d^2$$
$$d=\frac{L}{\sqrt{12}}$$

39. To determine the period, the angular frequency must be found with

$$a=-(\text{constant})x$$

Consider first the iceberg in equilibrium in water. Then, the gravity and buoyancy forces are acting on the iceberg. Therefore,

$$-mg+\rho_{water}gV_f=0$$

The mass of the iceberg is equal to its density multiplied by its volume and the volume of the submerged part of the iceberg is L^2x_0. (x_0 is the height of the part of the iceberg in the water.) The force equation then becomes

$$-mg+\rho_{water}gV_f=0$$
$$-\rho_{ice}L^3g+\rho_{water}gL^2x_0=0$$
$$\rho_{ice}gx_0=\rho_{water}Lg$$

Now, imagine that the iceberg moves a bit upwards by a distance x. The height of the submerged part then becomes x_0-x, and buoyant force declines. Thus, the balance of forces is broken and the iceberg accelerates. The force equation then becomes

$$-mg+\rho_{ice}gV_f=ma$$
$$-\rho_{ice}L^3g+\rho_{water}gL^2\left(x_0-x\right)=\rho_{ice}L^3a$$
$$-\rho_{ice}Lg+\rho_{water}g\left(x_0-x\right)=\rho_{ice}La$$
$$-\rho_{ice}Lg+\rho_{water}gx_0-\rho_{eau}gx=\rho_{ice}La$$

But since

$$\rho_{water}gx_0=\rho_{ice}Lg$$

We are left with

$$-\cancel{\rho_{ice} L g} + \cancel{\rho_{water} g x_0} - \rho_{water} g x = \rho_{ice} L a$$
$$-\rho_{water} g x = \rho_{ice} L a$$
$$a = -\frac{\rho_{water} g}{\rho_{ice} L} x$$

This is in agreement with the condition that must be respected to have a harmonic oscillation. This means that

$$\omega^2 = \frac{\rho_{water}}{\rho_{ice} L}$$
$$\omega^2 = \frac{1000 \frac{kg}{m^3} \cdot 9.8 \frac{N}{kg}}{920 \frac{kg}{m^3} \cdot 75m}$$
$$\omega^2 = 0.1420 \frac{rad^2}{s^2}$$
$$\omega = 0.3769 \frac{rad}{s}$$

Therefore, the period of oscillation is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{0.3769 s^{-1}} = 16.67 \text{ s}$$

40. If the force is

$$F = -cx^3$$

then the potential energy is

$$U = \frac{1}{4} c x^4$$

(Since $F = -dU / dx$)

According to energy conservation, we have

$$E_{mec} = \frac{1}{2} m v^2 + \frac{1}{4} c x^4$$

When the object is at its farthest point from the equilibrium position (so at $x = A$) there is no kinetic energy. Thus

$$E_{mec} = \frac{1}{4}cA^4$$

By combining these two equations, we obtain

$$\frac{1}{4}cA^4 = \frac{1}{2}mv^2 + \frac{1}{4}cx^4$$

If we solve for the speed in this equation, the result is

$$v = \sqrt{\frac{1}{2m}c\left(A^4 - x^4\right)}$$

With this formula, it is possible to calculate the time it takes to go from the equilibrium position to $x = A$. (This time is equal to one quarter of the period.)

The time to travel a small distance dx is

$$\begin{aligned} dt &= \frac{dx}{v} \\ &= \frac{dx}{\sqrt{\frac{1}{2m}c\left(A^4 - x^4\right)}} \\ &= \sqrt{\frac{2m}{c}}\frac{dx}{\sqrt{A^4 - x^4}} \end{aligned}$$

If all these time to travel from $x = 0$ to $x = A$ are added, the result is

$$\frac{T}{4} = \sqrt{\frac{2m}{c}}\int_0^A \frac{dx}{\sqrt{A^4 - x^4}}$$

We will then use new variables

$$\begin{aligned} x &= Au \\ dx &= Adu \end{aligned}$$

Thus, the integral becomes

$$\begin{aligned}\frac{T}{4} &= \sqrt{\frac{2m}{c}}\int_0^1 \frac{Adu}{\sqrt{A^4 - A^4u^4}} \\ &= \sqrt{\frac{2m}{c}}\int_0^1 \frac{Adu}{\sqrt{A^4\left(1-u^4\right)}} \\ &= \sqrt{\frac{2m}{c}}\int_0^1 \frac{Adu}{A^2\sqrt{\left(1-u^4\right)}} \\ &= \sqrt{\frac{2m}{c}}\frac{1}{A}\int_0^1 \frac{du}{\sqrt{\left(1-u^4\right)}}\end{aligned}$$

This leads to

$$T = 4\sqrt{\frac{2m}{c}}\frac{1}{A}\int_0^1 \frac{du}{\sqrt{\left(1-u^4\right)}}$$

That's it, it's done. We see that the period is proportional to 1/*A*. The integral does not change anything since the value of the integral does not depend on *A*. It's a simple number (actually, the value of this integral is 1.3110288).