11 CENTRE OF MASS

A 60 kg person is 60 cm from the left end of a 5 m long canoe having a mass of 90 kg. He then moves to the right end of the canoe to sit 60 cm from the other end of the canoe. By how much has the canoe shifted if there is no friction between the canoe and the water?

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Discover the answer to this question in this chapter.

Until now, the size of objects was not considered. It wasn't obvious then, but objects were considered as being only points. Of course, the forces were placed at the correct points of applications on the object, but this really had no influence whatsoever on our equations of forces. The size of objects will now be considered as this will have a significant influence in the next chapter.

It will also be shown that, even if the size of objects is taken into account, everything that was done in the previous chapters was correct. We were then simply describing the motion of the centre of mass of the object.

11.1 POSITION OF THE CENTRE OF MASS

Centre of Mass of a System of Particles

The position of the centre of mass is defined as

Centre of Mass of a System Composed of Particles

$$\vec{r}_{cm} = \frac{1}{m}\sum \vec{r}_i m_i$$

In components:

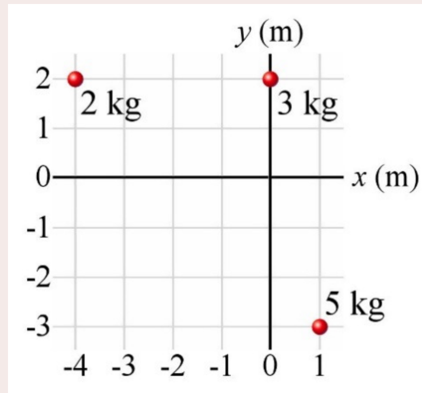
$$x_{cm} = \frac{1}{m}\sum x_i m_i \qquad y_{cm} = \frac{1}{m}\sum y_i m_i \qquad z_{cm} = \frac{1}{m}\sum z_i m_i$$

In these equations, m is the total mass of the system and $\sum x_i m_i$ is the sum of the x coordinate multiplied by the mass for each particle. The small i are there to identify the masses since a number is given to each mass of the system.

These formulas seem to come out of nowhere, but this is not the case. It will be shown later that they give the position of the point in the object whose motion is described by Newton's laws.

The centre of mass has been used for a very long time. Archimedes already used it in the 3rd century BC.

Example 11.1.1



Where is the centre of mass of these three particles?

The x-coordinate of the centre of mass is

$$\begin{aligned} x_{cm} &= \frac{1}{m}\sum xm \\ &= \frac{1}{m}\left(x_1 m_1 + x_2 m_2 + x_3 m_3\right) \\ &= \frac{1}{10kg}\left((-4m)\cdot 2kg + 0m\cdot 3kg + 1m\cdot 5kg\right) \\ &= \frac{-3kg\ m}{10kg} \\ &= -0.3m \end{aligned}$$

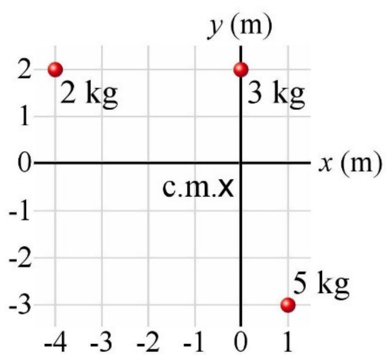
The y-coordinate of the centre of mass is

$$\begin{aligned} y_{cm} &= \frac{1}{m}\sum ym \\ &= \frac{1}{m}\left(y_1 m_1 + y_2 m_2 + y_3 m_3\right) \\ &= \frac{1}{10kg}\left(2m\cdot 2kg + 2m\cdot 3kg + (-3m)\cdot 5kg\right) \\ &= \frac{-5kg\ m}{10kg} \\ &= -0.5m \end{aligned}$$

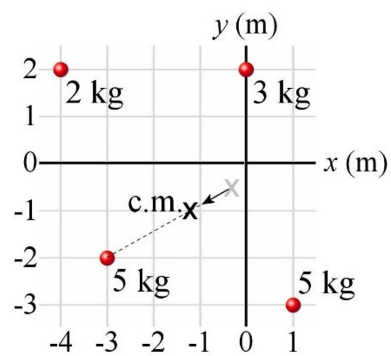
Therefore, the centre of mass is at the position (-0.3 m, -0.5 m).

Note that if mass is added to a system, the center of mass moves towards the added mass.

Before adding a mass

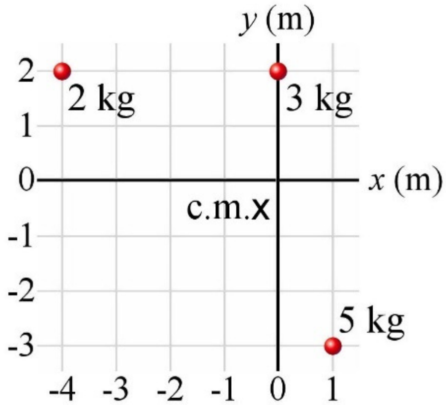


After adding a mass

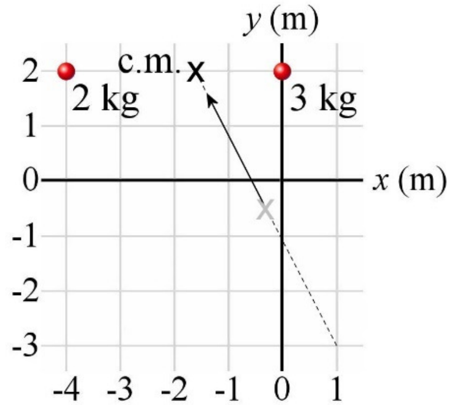


If mass is removed in a system, the center of mass moves in the opposite direction to the mass removed.

Before removing a mass



After removing a mass



Centre of Mass of an Object

Calculation With an Integral

To find the centre of mass of an object, the same formulas are used. However, to use them, the object must be divided into tiny particles. The formulas of the position of the centre of mass of a system of particles are then applied with all these particles.

However, the sums with infinitesimally small particles are

$$\lim_{m\to 0}\sum x_i m_i = \int x\,dm$$

$$\lim_{m\to 0}\sum y_i m_i = \int y\,dm$$

$$\lim_{m\to 0}\sum z_i m_i = \int z\,dm$$

and the formulas become

$$x_{cm} = \frac{1}{m}\int x\,dm \qquad y_{cm} = \frac{1}{m}\int y\,dm \qquad z_{cm} = \frac{1}{m}\int z\,dm$$

Linear Density, Surface Density and Volume Density

In the application of these formulas, three quantities are often used depending on the shape of the object.

a) Linear Density (λ)

Used for objects in one dimension (rods or wire), it indicates the mass per unit length. It is measured in *kg/m*. It can be calculated with

$$\lambda = \frac{mass}{length}$$

b) Surface Density (σ)

Used for objects in two dimensions (plates), it indicates the mass per unit surface. It is measured in *kg/m²*. It can be calculated with

$$\sigma = \frac{mass}{area}$$

c) Volume Density (or Density) (ρ)

Used for objects in three dimensions, it indicates the mass per unit volume. It is measured in *kg/m³*. It can be calculated with

$$\rho = \frac{mass}{volume}$$

When the object has a uniform density, the values of λ, σ and ρ (one of these 3 variables is used depending on the shape of the object) are constants. In this case, the following relationships can be used.

$$\lambda = \frac{total\ mass}{total\ length} \qquad \sigma = \frac{total\ mass}{total\ area} \qquad \rho = \frac{total\ mass}{total\ volume}$$

If the density of the object varies with position, then the values of λ, σ, and ρ are not constants, and the previous relationships cannot be used. The values of λ, σ and ρ must the be found by taking only very small pieces of the object whose mass is *dm*. In this case, we have

$$\lambda = \frac{dm}{dx} \qquad \sigma = \frac{dm}{dA} \qquad \rho = \frac{dm}{dV}$$

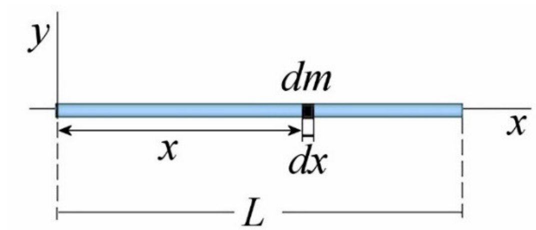
dx is the length of the small piece if it's a rod, *dA* is the area of the small piece if it's a plate, and *dV* is the volume of the small piece if it's a 3-dimensional object.

Calculating the Position of a Rod's Center of Mass

Most of the time, the calculation of the position of the centre of mass is very complex. In the worst cases, two double integrals must be calculated for a two-dimensional object, and three triple integrals must be calculated for a three-dimensional object. As you have never done any double and triple integrals (those who will take the *advanced calculus* course next year will see these concepts), that kind of calculation will not be done here. However, most of you can do the calculation in one dimension, i.e. for a rod.

To make this calculation, the rod is divided into small pieces of infinitesimal length. Each piece has a length *dx* and a mass *dm*. The linear density of one of these small pieces is

$$\lambda = \frac{dm}{dx}$$



The mass of the small piece is then

$$dm = \lambda\, dx$$

Then, the equation for the position of the centre of mass

$$x_{cm} = \frac{1}{m}\int x\, dm$$

becomes

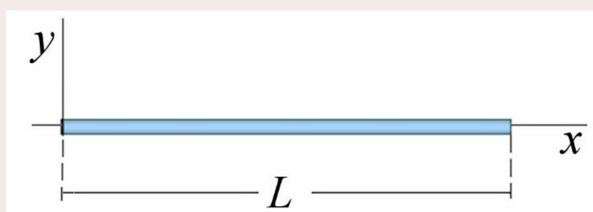
Centre of Mass of a Rod

$$x_{cm} = \frac{1}{m}\int \lambda x\, dx$$

It only remains to put the linear density in this formula. It may be constant, or it may vary depending on the position.

Example 11.1.2

Where is the centre of mass of a rod of constant linear density?



With the system of coordinates shown in the diagram, the rod is going from $x = 0$ to $x = L$. With a constant density, the position of the centre of mass is

$$\begin{aligned} x_{cm} &= \frac{1}{m}\int_0^L \lambda x\, dx \\ &= \frac{\lambda}{m}\int_0^L x\, dx \\ &= \frac{\lambda}{m}\left[\frac{x^2}{2}\right]_0^L \end{aligned}$$

$$= \frac{\lambda L^2}{2m}$$

Since the density is uniform, we have

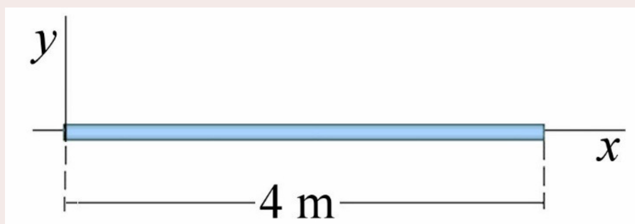
$$\lambda = \frac{total\ mass}{total\ length} = \frac{m}{L}$$

The mass of the rod is thus $m = \lambda L$. Therefore, the position of the centre of mass is

$$\begin{aligned} x_{cm} &= \frac{\lambda L^2}{2m} \\ &= \frac{\lambda L^2}{2\lambda L} \\ &= \frac{L}{2} \end{aligned}$$

This indicates that the centre of mass of the rod is at the midpoint of the rod.

Example 11.1.3



Where is the centre of mass of a 4 m long rod whose linear density is given by the formula $\lambda = 3\frac{\text{kg}}{\text{m}^2}\cdot x + 6\frac{\text{kg}}{\text{m}}$ if the x-axis is as shown in the diagram?

The rod goes from $x = 0$ to $x = 4$ m. The integral is thus

$$\begin{aligned} x_{cm} &= \frac{1}{m}\int_0^{4\,\text{m}} \lambda x\, dx \\ &= \frac{1}{m}\int_0^{4\,\text{m}} \left(3\tfrac{\text{kg}}{\text{m}^2}\cdot x + 6\tfrac{\text{kg}}{\text{m}}\right) x\, dx \\ &= \frac{1}{m}\int_0^{4\,\text{m}} \left(3\tfrac{\text{kg}}{\text{m}^2}\cdot x^2 + 6\tfrac{\text{kg}}{\text{m}}\cdot x\right) dx \\ &= \frac{1}{m}\left[1\tfrac{\text{kg}}{\text{m}^2}\cdot x^3 + 3\tfrac{\text{kg}}{\text{m}}\cdot x^2\right]_0^{4\,\text{m}} \\ &= \frac{1}{m}\left[1\tfrac{\text{kg}}{\text{m}^2}\cdot (4\text{ m})^3 + 3\tfrac{\text{kg}}{\text{m}}\cdot (4\text{ m})^2\right] \\ &= \frac{112\text{ kgm}}{m} \end{aligned}$$

The mass of the rod is needed. This mass cannot ne found by simply calculating $m = \lambda L$ since the linear density is not constant. When the linear density is not a constant, we start by finding the mass of a small piece

$$\lambda = \frac{dm}{dx}$$
$$dm = \lambda dx$$

then we find the total mass by adding up all the masses *dm*.

$$m = \int \lambda dx$$

This formula is always valid for rods of variable linear density. Therefore, the mass is

$$\begin{aligned} m &= \int_0^{4\text{ m}} \lambda \, dx \\ &= \int_0^{4\text{ m}} \left(3\tfrac{\text{kg}}{\text{m}^2} \cdot x + 6\tfrac{\text{kg}}{\text{m}}\right) dx \\ &= \left[3\tfrac{\text{kg}}{\text{m}^2} \cdot \frac{x^2}{2} + 6\tfrac{\text{kg}}{\text{m}} \cdot x\right]_0^{4\text{ m}} \\ &= \left[3\tfrac{\text{kg}}{\text{m}^2} \cdot \frac{(4\text{ m})^2}{2} + 6\tfrac{\text{kg}}{\text{m}} \cdot 4\text{ m}\right] \\ &= 48\text{ kg} \end{aligned}$$

Thus, the position of the centre of mass is

$$\begin{aligned} x_{cm} &= \frac{112\text{ kgm}}{m} \\ &= \frac{112\text{ kgm}}{48\text{ kg}} \\ &= 2.333\text{m} \end{aligned}$$

The centre of mass is not at the midpoint of the rod anymore (which is at $x = 2$ m), but is a little more towards the right end side of the rod. This seems logic because the rod is denser to the right. The right side of the rod is, therefore, more massive than the left side and it is normal to have a centre of mass shifted a little towards the right compared to the position of the centre of mass of the uniform rod.

Use of Symmetry to Find the Centre of Mass of an Object

It was found that the centre of mass of a uniform rod is at the midpoint of the rod. It could have been predicted that the centre of mass must be at the midpoint because the rod is symmetrical. If the centre of mass were not at the midpoint, it would break the symmetry. As both sides of the rod are identical, there is no reason for the centre of mass to be more on one side than on the other.

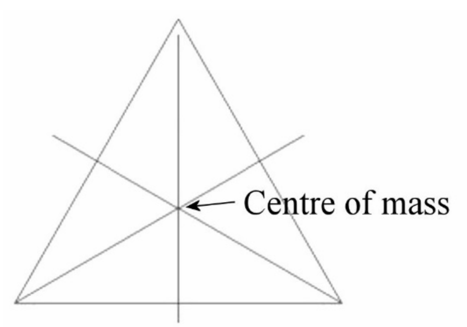
Actually, the centre of mass of an object must be on the axis of symmetry if the object has an axis of symmetry. If there is more than one axis, the centre of mass is at the intersection of the axes of symmetry. It is impossible for the axes of symmetry to not all intersect at the same point.

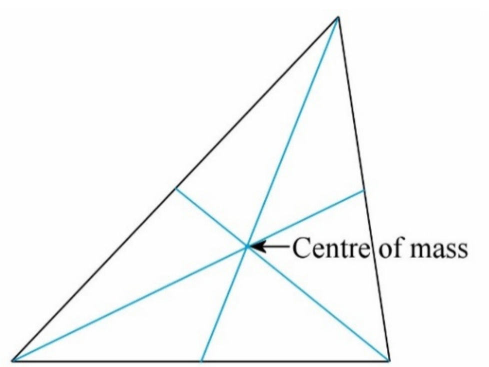
Centre of Mass and Axes of Symmetry

If the object has an axis of symmetry, the centre of mass must be on this axis. If there are several axes of symmetry, the centre of mass is at the intersection of the axes.

The shape of the object is not the only thing that must be symmetrical; the density of the object must also be. The rod with a varying density was an example of that. The object was symmetrical, but its density was larger on one side. This broke the symmetry.

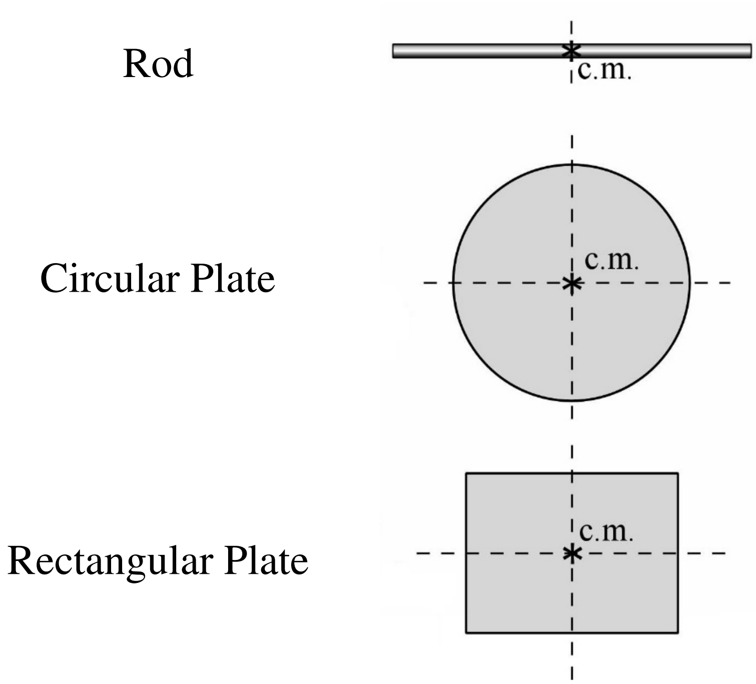
This simple use of symmetry, therefore, allows us to find the centre of mass of simple objects without having to make long calculations. Take a triangular plate, for example. There are three axes of symmetry on this triangular plate. The centre of mass is at the intersection of these axes.





(Here's a small mathematical note: the centre of mass of a triangular plate is always at the point of intersection of the medians of the triangle. This method is even more general because the centre of mass can then be found even if the triangle is not symmetrical. For those who do not know what a median is, it is a line that goes from a summit to the midpoint of the opposite side.)

Using this trick, the centre of mass of many objects can be found. Only rods and plates will be considered here, but this idea could be applied to three-dimensional objects. (The centre of mass of three-dimensional objects is at the intersection of the planes of symmetry.)



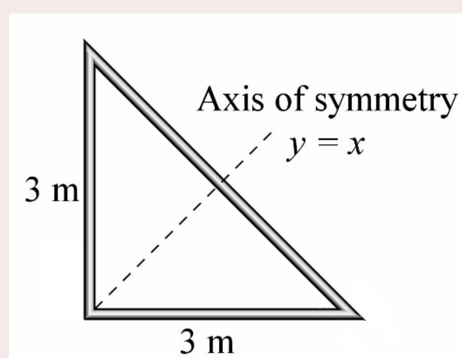
The principles given earlier remain valid: if mass is added, the center of mass moves towards the added mass and if mass is removed, the center of mass moves in the opposite direction to the removed mass.

When dealing with an object that is not symmetrical but composed of symmetrical objects, the centre of mass can be found by using the following trick:

> If an object consists of parts for which the position of the centre of mass is known, each part can be replaced by a point mass located at the centre of mass of the part. Then apply the formulas to find the centre of mass of a system consisting of particles to find the centre of mass of the object.

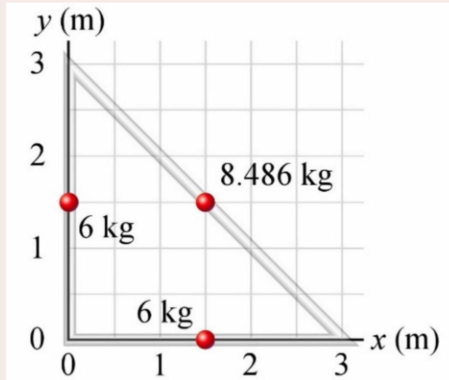
Example 11.1.4

Where is the centre of mass of this assemblage of 3 rods? The linear density of each rod is 2 kg/m.



> It can be noted first that there is an axis of symmetry at 45°. This axis has the equation $y = x$. Therefore, there is no need to calculate the position of the centre of mass for the two coordinates. When the x-coordinate of the centre of mass is found, the y-coordinate will be known because both must be equal.

As it is known that the centre of mass of a uniform rod is at the midpoint of the rod, each rod can be replaced by a point mass located at the midpoint of the rod. The mass of both 3 m rods is 2 kg/m · 3 m = 6 kg while the mass of the rod which forms the hypotenuse is 2 kg/m · 4.243 m = 8.486 kg. The situation shown in the diagram to the right is thus obtained.



You might wonder how the midpoint of the rod which forms the hypotenuse was found. In fact, it is pretty easy. For the x-coordinate, one of the ends of the rod is at $x = 0$ and the other end is at $x = 3$ m. The midpoint is, therefore, at $x = 1.5$ m. For the y-coordinate, one of the ends of the rod is at $y = 0$ and the other end is at $y = 3$ m. The midpoint is, therefore, at $y = 1.5$ m.

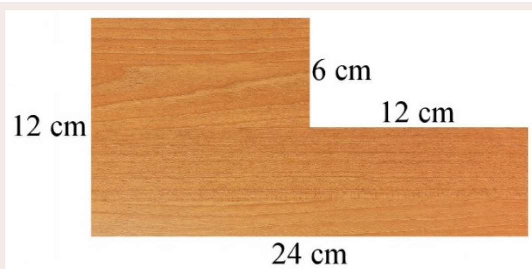
The formula of the centre of mass is then applied.

$$\begin{aligned} x_{cm} &= \frac{1}{m}\sum x_i m_i \\ &= \frac{1}{20.486kg}\left(0m \cdot 6kg + 1.5m \cdot 6kg + 1.5m \cdot 8.486kg\right) \\ &= \frac{21.729kg\ m}{20.486kg} \\ &= 1.0607m \end{aligned}$$

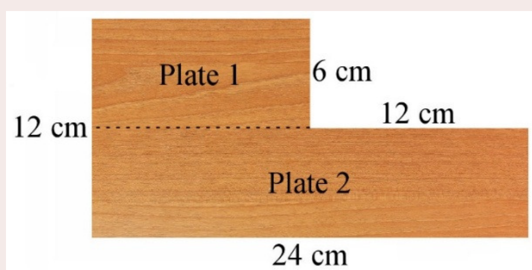
Therefore, the centre of mass is at the position (1.0607 m, 1.0607 m).

Example 11.1.5

Where is the centre of mass of this wooden plate if the surface density is constant?

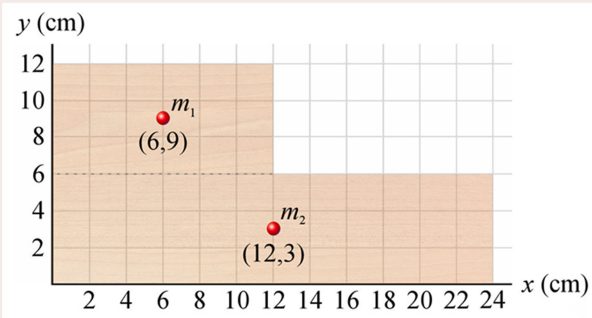


There is no axis of symmetry here. The position of the centre of mass must, therefore, be calculated for both coordinate x and y. To achieve this, this plate is separated into two rectangular plates.



As it is known that the centre of mass of a uniform rectangular plaque is at the centre of the plate, each plate is replaced by a point mass located at the centre of the plate. The value of the density is not given, but this has no impact at the end.

The mass of each plate is



$$m_1 = \sigma \cdot area = \sigma \cdot 72cm^2$$
$$m_2 = \sigma \cdot area = \sigma \cdot 144cm^2$$

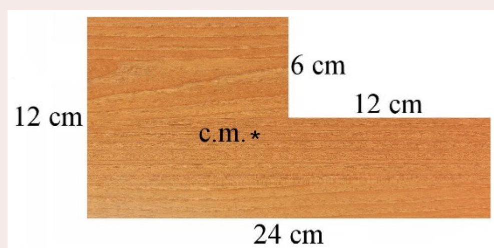
The diagram shows the position of these two point masses.

The formulas for the position of the centre of mass give

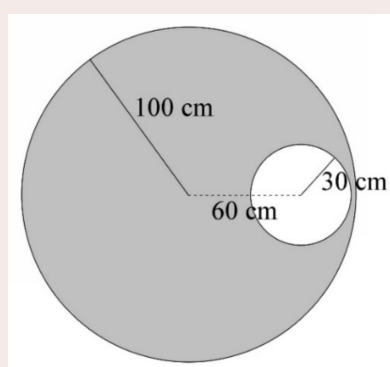
$$\begin{aligned} x_{cm} &= \frac{1}{m}\sum x_i m_i \\ &= \frac{6cm \cdot \sigma \cdot 72cm^2 + 12cm \cdot \sigma \cdot 144cm^2}{\sigma \cdot 72cm^2 + \sigma \cdot 144cm^2} \\ &= \frac{\cancel{\sigma} \cdot 2160cm^3}{\cancel{\sigma} \cdot 216cm^2} \\ &= 10cm \end{aligned}$$

$$\begin{aligned} y_{cm} &= \frac{1}{m}\sum y_i m_i \\ &= \frac{9cm \cdot \sigma \cdot 72cm^2 + 3cm \cdot \sigma \cdot 144cm^2}{\sigma \cdot 72cm^2 + \sigma \cdot 144cm^2} \\ &= \frac{\cancel{\sigma} \cdot 1080cm^3}{\cancel{\sigma} \cdot 216cm^2} \\ &= 5cm \end{aligned}$$

Therefore, the centre of mass is at (10 cm, 5 cm)



Example 11.1.6



Where is the centre of mass of this metal plate in which there is a hole if the surface density is constant?

An x and a y-axis whose origin is at the centre of the circular plate is used.

As there is an axis of symmetry (a horizontal axis passing through the centre of the circular plate), the y-coordinate of the centre of mass must be y_{cm} = 0 m.

It is harder to find the x-coordinate. As the plate cannot be divided into circular or rectangular pieces, another trick must be found.

This trick is to consider a circular plate <u>without a hole</u> formed of two plates:

1) A circular plate with a hole (index 1 refers to this plate)
2) A smaller circular plate which would plug the hole (index 2 refers to this plate)

The following formula then gives the position of the centre of mass of this plate without a hole.

$$x_{cm} = \frac{1}{m}\sum x_i m_i$$

$$x_{cm} = \frac{m_1 \cdot x_{1cm} + m_2 \cdot x_{2cm}}{m_{\text{plate without hole}}}$$

Obviously, the position of the centre of mass of the plate without a hole is at the centre of the plate, so at x_{cm}= 0. The equation then becomes

$$0 = \frac{m_1 \cdot x_{1cm} + m_2 \cdot x_{2cm}}{m_{\text{plate without hole}}}$$

$$0 = m_1 \cdot x_{1cm} + m_2 \cdot x_{2cm}$$

The position of the centre of mass of the plate with a hole can then be found if this equation is solved for x_{1cm}.

$$0 = m_1 \cdot x_{1cm} + m_2 \cdot x_{2cm}$$

$$m_1 \cdot x_{1cm} = -m_2 \cdot x_{2cm}$$

$$x_{1cm} = \frac{-m_2 \cdot x_{2cm}}{m_1}$$

The centre of mass of the plate that would plug the hole is at the centre of the hole, so at x_{2cm} = 0.6 m.

As in the previous example, the masses are found by multiplying the surface density by the area of the plate. The masses are then

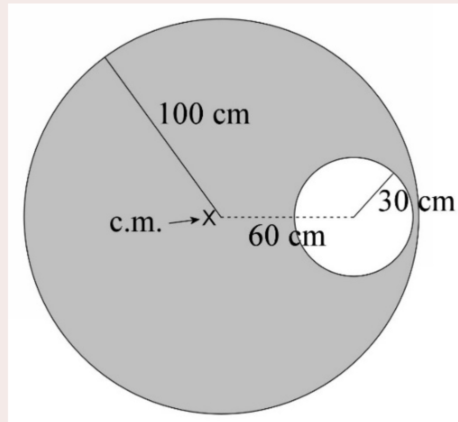
$$m_2 = \sigma \cdot \pi \cdot (0.3m)^2$$

$$m_1 = \sigma \cdot \left[\pi \cdot (1m)^2 - \pi \cdot (0.3m)^2\right]$$

Therefore,

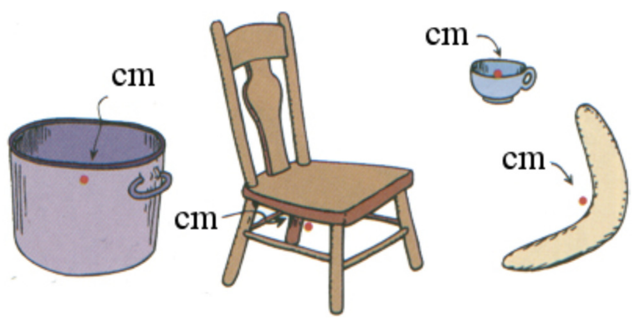
$$x_{1cm} = \frac{-m_2 \cdot x_{2cm}}{m_1}$$
$$= \frac{-\sigma \cdot \pi \cdot (0.3m)^2 \cdot 0.6m}{\sigma \cdot \left[\pi \cdot (1m)^2 - \pi \cdot (0.3m)^2\right]}$$
$$= \frac{-(0.3m)^2 \cdot 0.6m}{(1m)^2 - (0.3m)^2}$$
$$= -0.05934m$$

The centre of mass is thus 5.934 cm to the left of the centre of the plate.



This result agrees with the principle given earlier which says that if mass is removed from a system, the center of mass moves in the opposite direction to the mass removed. Here, the center of mass would have been exactly in the center of the plate if it weren't for this hole. If mass is removed on the right to make the hole, then the center of mass moves to the left (opposite direction).

Note that the centre of mass is not necessarily within the material that makes up the object. The image shows the position of the centre of mass of 4 items. For all these objects, the centre of mass is outside of material that makes up the object, which means that it is possible to touch the centre of mass with a finger. (You would feel absolutely nothing if you were to touch these centres of mass.)



schools.wikia.com/wiki/Center_of_Mass

11.2 SOME IMPORTANT RESULTS CONCERNING THE CENTRE OF MASS

Gravitation and Centre of Mass

Gravitational Force

Let's begin with a reminder of the force of gravitation.

Weight or Gravitational Force (formula valid near the surface of the Earth)

1) Magnitude of the force

$$F_g = mg$$

2) Direction of the force

Downwards (Towards the centre of the Earth)

3) Application point of the force

From the centre of mass of the object.

The fact that the point of application of the weight is at the centre of mass is the first reason why the centre of mass is important. Actually, the force of gravitation is exerted on every atom of the object, but the equations of motion are the same if it is assumed that all the force is exerted on the centre of mass. This will be demonstrated in the next chapter.

(More precisely, the point of application of weight is at the centre of gravity of the object. This centre can be at a different location than the centre of mass if the gravitational acceleration is not the same everywhere in the object. This is the case for an enormous object, like a mountain. Then, the centre of gravity is a little below the centre of mass because the gravitational acceleration is slightly lower at the top of the mountain than at the base of the mountain. This small difference will be ignored here.)

Gravitational Energy

The centre of mass also solves a difficult problem: what height should be used to calculate the gravitational energy of an object? Which value of y should be used in mgy for a rod standing vertically on the ground, for example? Should the position of the lower end of the rod be used? Should the position of the higher end of the rod be used? Should the position of the middle of the rod be used? Actually, the total gravitational energy is the sum of the gravitational energy of each atom. Let's find out what the result of this sum is.

The energy of one atom is

$$U_{gi} = m_i g y_i$$

The total gravitational energy of the object is the sum of these energies.

$$\begin{aligned} U_g &= \sum \left(m_i g y_i \right) \\ &= \sum \left(m_i y_i \right) g \end{aligned}$$

But since

$$y_{cm} = \frac{\sum m_i y_i}{m}$$
$$\sum m_i y_i = m y_{cm}$$

the gravitational energy is

$$U_g = \sum (m_i y_i) g$$
$$= m y_{cm} g$$

Thus, the sum of the energy of each atom gives exactly the same result as if energy is calculated by assuming that all the mass is concentrated at the centre of mass.

Centre of Mass and Gravitational Energy

When calculating the gravitational energy of an object, the position of the centre of mass must be used for *y* in *mgy*.

Total Momentum of a System

Consider the definition of the position of the centre of mass in vector form.

$$\vec{r}_{cm} = \frac{1}{m} \sum \vec{r}_i m_i$$

(If this equation is resolved into components, the 3 equations for the position of the centre of mass in *x*, *y* and *z* are obtained.)

If both sides of the equation are derived, it becomes

$$m\vec{r}_{cm} = \sum \vec{r}_i m_i$$
$$\frac{d(m\vec{r}_{cm})}{dt} = \frac{d\left(\sum \vec{r}_i m_i\right)}{dt}$$
$$m\frac{d(\vec{r}_{cm})}{dt} = \sum m_i \frac{d\vec{r}_i}{dt}$$
$$m\vec{v}_{cm} = \sum m_i \vec{v}_i$$

The term on the right represents the total momentum of the system. Therefore

Total Momentum of a System

$$\vec{p}_{tot} = m\vec{v}_{cm}$$

If the momentum of a block must be calculated, it is pointless to calculate the momentum of every atom in order to sum them (with a vector sum!). It can be calculated by simply multiplying the mass of the block by the velocity of its centre of mass. This gives the same result as the vector sum of the momentum of all the atoms, even if the block spins. This is how the momentum was calculated in the previous chapter.

When the centre of mass is moving, the velocity of the centre of mass might be needed to solve a problem. This velocity can be obtained with the equation obtained previously.

$$m\vec{v}_{cm} = \sum m_i \vec{v}_i$$
$$\vec{v}_{cm} = \frac{1}{m}\sum m_i \vec{v}_i$$

Thus, the result is

Velocity of the Centre of Mass

$$\vec{v}_{cm} = \frac{1}{m}\sum m_i \vec{v}_i$$

In components:

$$v_{cm\,x} = \frac{1}{m}\sum m_i v_{xi} \qquad v_{cm\,y} = \frac{1}{m}\sum m_i v_{yi} \qquad v_{cm\,z} = \frac{1}{m}\sum m_i v_{zi}$$

Newton's First Law

If the momentum equation previously obtained is derived once again, it becomes

$$m\vec{v}_{cm} = \sum m_i \vec{v}_i$$
$$\frac{d\left(m\vec{v}_{cm}\right)}{dt} = \frac{d\left(\sum m_i \vec{v}_i\right)}{dt}$$
$$m\frac{d\left(\vec{v}_{cm}\right)}{dt} = \sum m_i \frac{d\vec{v}_i}{dt}$$
$$m\vec{a}_{cm} = \sum m_i \vec{a}_i$$

Since $\Sigma F = ma$ for each atom, then the sum is

$$m\vec{a}_{cm} = \sum \vec{F}_i$$

The forces on each atom are of two types: internal forces and external forces.

$$m\vec{a}_{cm} = \sum \vec{F}_{i\,\text{int}} + \sum \vec{F}_{i\text{ext}}$$

It has already been demonstrated in the previous chapter that the sum of the internal forces is always zero because of Newton's third law. Therefore,

$$m\vec{a}_{cm} = \cancel{\sum \vec{F}_{i\,\text{int}}} + \sum \vec{F}_{i\,ext}$$

If the sum of the external forces is zero, Newton's first law is obtained

Newton's First Law

$$m\vec{a}_{cm} = 0 \qquad \text{if} \qquad \sum \vec{F}_{ext} = 0$$

The law of inertia is obtained again. This law means that an object cannot change his own motion. The internal forces, made by the atoms of the object itself, cannot alter the velocity of an object because they always cancel out. The force must be exerted by another object. In the absence of such an external force, the velocity of the object cannot change.

Note that if Newton's third law were to be false, then the internal forces would not cancel out and internal forces could set the object in motion, which would contradict Newton's first law. Therefore, Newton's third law must be true for Newton's first law to be true.

This is a more precise version of the Newton's first law. It says that the velocity of the centre of mass must remain constant when there are no external forces. For an astronaut at rest in space, far away from any other masses, there is no external forces and its centre of mass must always remain in the same place. There is no movement that the astronaut can do to set its centre of mass in motion since these movements are all internal forces that cannot change the velocity of the centre of mass. If he were to throw something, say a boot, he would start to move in the opposite direction, but the centre of mass of the boot and the astronaut would remain at the same place.

Even if the centre of mass is moving at a constant velocity, the object can rotate. For example, the motion of this tool is made in the absence of external force.

schools.wikia.com/wiki/Center_of_Mass

The only point of the object which moves in a straight line is the centre of mass, as specified by Newton's first law.

All the other atoms in the object move in a more complicated path than a straight line.

Returning to the astronaut example, this means that the astronaut's movements may not change the velocity of its centre of mass, but they can make him rotate.

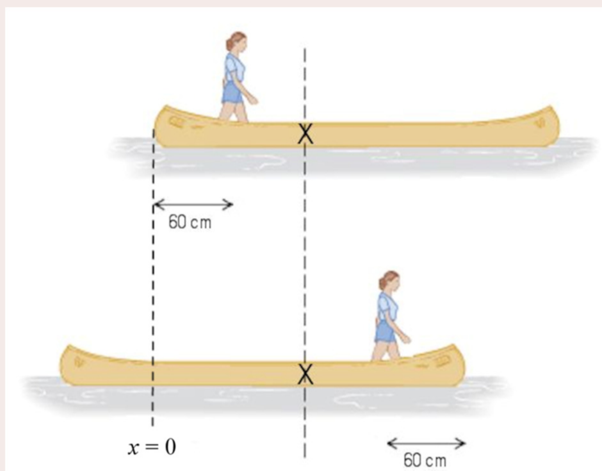
Movie Mistake

It is quite easy to find errors in science fiction movies, especially when a moving spaceship explodes. As the explosion is an internal force, the velocity of the centre of mass must remain the same. Therefore, the centre of mass of all the fragments should continue with the same velocity as the velocity of the spaceship before the explosion. In movies, however, the velocity of the centre of mass of the fragments often changes (most of the time the centre of mass stops) after the explosion. Look at this exploding spaceship in "Star Wars I".
https://physique.merici.ca/mecanique/explosion.wmv
When the ship explodes, the centre of mass of the fragments should continue with the same velocity and the fragments should strike the attacking vessel. The attacking ship should then be bombarded with many fragments from the exploding ship and many of these fragments could do much damage.

Here is an example of the application of this better version of Newton's first law.

Example 11.2.1

A 60 kg person is 60 cm from the left end of a 5 m long canoe having a mass of 90 kg. He then moves to the right end of the canoe to sit 60 cm from the other end of the canoe. By how much has the canoe shifted if there is no friction between the canoe and the water?



www.chegg.com/homework-help/questions-and-answers/450-woman-stands-600-canoe-500-long-walks-point-100-end-point-100-end-figure-intro-1-figur-q370549

The system consists here of the canoe and the person. The speed of the centre of mass is zero initially. As the sum of the external forces is zero, the speed of the centre of mass always remains zero. If the speed of the centre of mass is always zero, then the centre of mass must always remain at the same place when the person moves in the canoe. This is the main idea to solve this problem: The centre of mass stays at the same place in this situation.

The origin $x = 0$ is placed at the left end of the canoe when the person is at the left end of the canoe (see diagram).

Initially (top diagram), the centre of mass of the canoe is at x = 2.5 m and the person is at x = 0.6 m. Then, the position of the centre of mass of the person - canoe system is

$$\begin{aligned} x_{cm} &= \frac{1}{m}\sum x_i m_i \\ &= \frac{1}{150kg}\left(0.6m\cdot 60kg + 2.5m\cdot 90kg\right) \\ &= 1.74m \end{aligned}$$

If the canoe position were fixed, the centre of mass of the canoe would still be at x = 2.5 m after the change in position and the person would be x = 4.4 m. However, the position is not fixed, and the canoe moved a distance d towards the left. This shift of d moves the position of the centre of mass of the canoe to x = 2.5 m - d and the position the person at the end of the canoe to x = 4.4 m - d.

The position of the centre of mass is now

$$\begin{aligned} x'_{cm} &= \frac{1}{m}\sum x_i m_i \\ &= \frac{1}{150kg}\left(\left(4.4m - d\right)\cdot 60kg + \left(2.5m - d\right)\cdot 90kg\right) \\ &= \frac{489kg\,m - 150kg\cdot d}{150kg} \\ &= 3.26m - d \end{aligned}$$

The centre of mass stays at the same place. Therefore,

$$\begin{aligned} x_{cm} &= x'_{cm} \\ 1.74m &= 3.26m - d \\ d &= 1.52m \end{aligned}$$

The canoe thus moved 1.52 m towards the left.

You should be aware of this kind of displacement, especially if you are trying to disembark from a boat. It could be dangerous if you walk towards the front of the boat to disembark when the front touch at the wharf. As you walk forward, the boat moves backwards and the gap between the dock and the boat increases. When you arrive at the front of the boat, this gap will perhaps be too large to cross, and you may end up in the water.

Newton's Second Law

Let's return to the equation

$$m\vec{a}_{cm} = \cancel{\sum \vec{F}_{i\,\text{int}}} + \sum \vec{F}_{i\,ext}$$

If there are external forces, this equation is Newton's second law.

Newton's Second Law

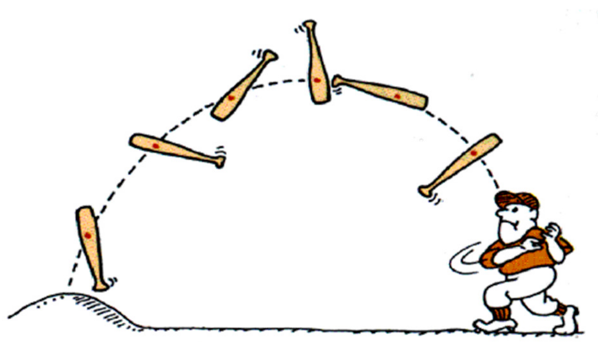
$$\sum \vec{F}_{ext} = m\vec{a}_{cm}$$

or

$$\sum \vec{F}_{ext} = \frac{d\vec{p}_{tot}}{dt}$$

These results show that Newton's second law actually allows us to predict the motion of the centre of mass.

For example, when we were saying earlier that a projectile follows a parabolic trajectory, we were really saying that the centre of mass of the projectile follows a parabolic trajectory.



www.ux1.eiu.edu/~addavis/1350/09Mom/CoM.html

The parabolic trajectory followed by the centre of mass of several projectile can be seen in this video.
https://www.youtube.com/watch?v=DY3LYQv22qY

This shows quite clearly that external forces are the only ones that can change the motion of the centre of mass. The speed of the centre of mass can be changed with an internal force.

The image on the right comes from a video showing several people pushing a pickup truck.
https://www.youtube.com/watch?v=zsR2jqRhT5E
Of course, the efforts of the person (let's call him Bob) in the pickup truck are useless. As he is part of the truck+Bob system, the forces exerted by Bob are internal forces and they cannot change the motion of the truck.

If the truck is the only object in the system considered, then the forces made by Bob are not the internal forces. However, the force exerted by Bob on the truck is cancelled by the force of friction exerted by Bob's feet. So, the total force exerted by Bob on the truck is zero and we still conclude that Bob's efforts are useless.

Which version of Newton's second law is more general?

We have two versions of the second law

$$\sum \vec{F}_{ext} = m\vec{a}_{cm} \qquad \sum \vec{F}_{ext} = \frac{d\vec{p}_{tot}}{dt}$$

and one may wonder which one is more general. We can often read that the second is more general since it gives to the first only if the mass is constant (as it was shown in the previous chapter). The second version would therefore be more general since it could apply to cases where the mass changes while the first could not.

However, this is not true. In a closed system, the mass cannot vary, and the first equation is as general as the second. In an open system, then there are some subtleties that make it so that, very often, neither of these two equations can be used without a small modification. For a more in-depth discussion of open systems, see this document.
https://physique.merici.ca/mechanics/TwoversionsNewton.pdf

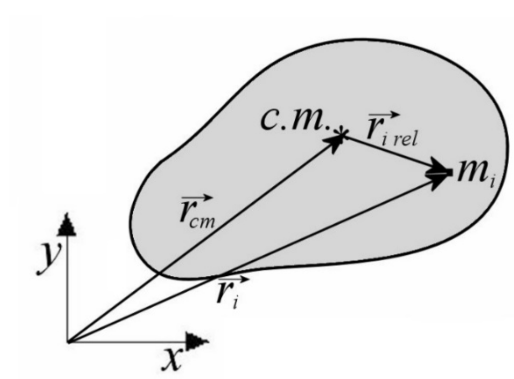
Total Kinetic Energy of a System

Since the total momentum of a system can be calculated by considering only the motion of the centre of mass with the formula

$$\vec{p}_{tot} = m\vec{v}_{cm}$$

we have to wonder if the same thing can be done for the total kinetic energy of a system. Can the total kinetic energy of a system be calculated with this simple formula?

$$E_{k\,tot} \overset{?}{=} \frac{1}{2}mv_{cm}^2$$



To answer this question, let's calculate the kinetic energy of a single atom in the object and then sum all these kinetic energies to get the total kinetic energy of the object.

The position of the atom can be written as

$$\vec{r} = \vec{r}_{cm} + \vec{r}_{rel}$$

If this formula is derived, it becomes

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}_{cm}}{dt} + \frac{d\vec{r}_{rel}}{dt}$$
$$\vec{v} = \vec{v}_{cm} + \vec{v}_{rel}$$

The last term in this equation is the difference in velocities between the atom and the centre of mass. It is called the relative velocity. If the atom has the same velocity (magnitude and direction) as the centre of mass, then v_{rel} vanishes.

The total kinetic energy of the system is then

$$\begin{aligned} E_{k\,tot} &= \sum \frac{1}{2} m_i v_i^2 \\ &= \sum \frac{1}{2} m_i \left(\vec{v}_{cm} + \vec{v}_{i\,rel} \right)^2 \\ &= \sum \frac{1}{2} m_i \left(v_{cm}^2 + 2\vec{v}_{cm} \cdot \vec{v}_{i\,rel} + v_{i\,rel}^2 \right) \\ &= \sum \frac{1}{2} m_i v_{cm}^2 + \sum m_i \vec{v}_{cm} \cdot \vec{v}_{i\,rel} + \sum \frac{1}{2} m_i v_{i\,rel}^2 \end{aligned}$$

The second term is

$$\begin{aligned} \sum m_i \vec{v}_{cm} \cdot \vec{v}_{i\,rel} &= \vec{v}_{cm} \cdot \sum m_i \vec{v}_{i\,rel} \\ &= \vec{v}_{cm} \cdot m\vec{v}_{cm\,rel} \end{aligned}$$

The fact that the net momentum can be found with the velocity of the centre of mass was used. But since v_{rel} measures the velocity difference between an object and the centre of mass, $v_{cm\,rel}$ is the velocity difference between the centre of mass and the centre of mass! Obviously, this speed is zero. The second term is, therefore, zero and the energy is

$$\begin{aligned} E_{k\,tot} &= \sum \frac{1}{2} m_i v_{cm}^2 + \sum \frac{1}{2} m_i v_{i\,rel}^2 \\ &= \frac{1}{2} \left(\sum m_i \right) v_{cm}^2 + \sum \frac{1}{2} m_i v_{i\,rel}^2 \end{aligned}$$

Finally, the energy is

Total Kinetic Energy of a System

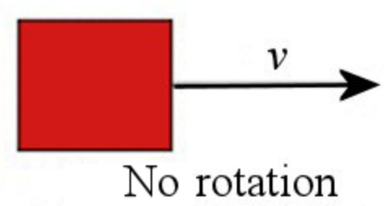
$$E_{k\,tot} = \frac{1}{2} m v_{cm}^2 + \sum \frac{1}{2} m_i v_{i\,rel}^2$$

The first term is called the *kinetic energy of the centre of mass* while the second term is known as the *kinetic energy relative to the centre of mass.*

Therefore, the kinetic energy of a system **cannot,** most of the time, be found by considering only the motion of the centre of mass as it can be done for momentum.

An Example: Kinetic Energies for Non-Rotating and Rotating Objects

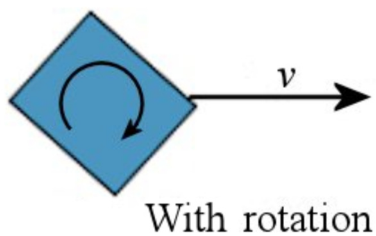
If an object is moving without rotating, all its atoms have the same velocity as the centre of mass. Then, all the v_{rel} are zero (since there is no difference in velocities between the atoms and the centre of mass) and the energy is simply



$$E_{k\,tot} = \frac{1}{2} m v_{cm}^2$$

The kinetic energy calculations made in the previous chapters were thus correct since all atoms of the objects were going at the same velocity as the centre of mass in all the calculations made so far.

But if the object is rotating while it is moving, then some atoms are going faster than the centre of mass (such as those above the centre of mass in our diagram) whereas some are going slower than the centre of mass (such as those below the centre of mass in our diagram). For momentum, these differences eventually cancelled out, but they do not for kinetic energy. Then, the kinetic energy is



$$E_{k\,tot} = \frac{1}{2} m v_{cm}^2 + \sum \frac{1}{2} m_i v_{i\,rel}^2$$

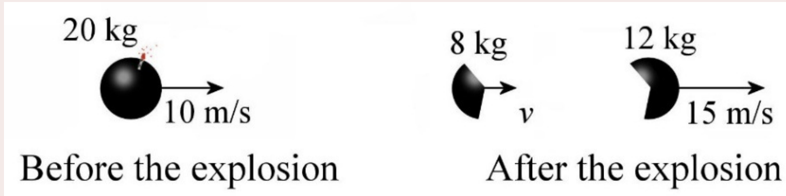
As the second term of this equation must be positive, the kinetic energy of a rotating object is larger than the energy of an object that is not rotating even if their masses and their speeds are the same. You might think that the calculation of the second term will be very long because it is a sum over all the atoms that make up the object. Fortunately, there are ways to calculate this term without doing this long sum. The methods used to do this when the object is rotating will be seen in the next chapter.

Other Possibilities

Rotation is not the only thing that can change the kinetic energy of the object. Everything that makes the velocity of an atom different from the velocity of the centre of mass changes the kinetic energy. For example, if the object is vibrating, the kinetic energy it would change. Thermal agitation of the atoms can also alter the velocity of the atoms relative to the centre of mass.

Example 11.2.2

A bomb explodes into two fragments as shown in the diagram.



a) What is the speed of the 8 kg fragment?

This speed can be found using the momentum conservation law or by using the fact that the velocity of the centre of mass after the explosion must be the same as it was before the explosion. As some examples of the application of the law of conservation of *p* were done in the previous chapter, this problem will be solved using the constancy of the velocity of the centre of mass.

Before the explosion, the velocity of the centre of mass is obviously 10 m/s.

$$v_{cm} = 10\tfrac{m}{s}$$

After the explosion, the velocity of the centre of mass is calculated from the velocity of each fragment.

$$\begin{aligned} v'_{cm} &= \frac{1}{m}\sum m_i v'_i \\ &= \frac{1}{20kg}\left(12kg \cdot 15\tfrac{m}{s} + 8kg \cdot v\right) \end{aligned}$$

As the velocity of the centre of mass does not change, we have

$$\begin{aligned} v_{cm} &= v'_{cm} \\ 10\tfrac{m}{s} &= \frac{1}{20kg}\left(12kg \cdot 15\tfrac{m}{s} + 8kg \cdot v\right) \\ v &= 2.5\tfrac{m}{s} \end{aligned}$$

b) What is the kinetic energy of the system after the explosion?

This calculation will be made using two different methods. In the first method, the kinetic energy of each fragment is added.

$$\begin{aligned} E_{k\,tot} &= \sum \frac{1}{2} m_i v_i^2 \\ &= \frac{1}{2}\cdot 12kg \cdot \left(15\tfrac{m}{s}\right)^2 + \frac{1}{2}\cdot 8kg \cdot \left(2.5\tfrac{m}{s}\right)^2 \\ &= 1375J \end{aligned}$$

The second method is done here mainly to illustrate how to use v_{rel} in the following energy formula.

$$E_{k\,tot} = \frac{1}{2}mv_{cm}^2 + \sum \frac{1}{2}mv_{i\,rel}^2$$

The first fragment (the 12 kg fragment) has a velocity of 15 m/s. As the centre of mass has a velocity of 10 m/s, the difference in velocities between the fragment and the centre of mass is 5 m/s.

$$v_{1\,rel} = 5\tfrac{m}{s}$$

The second fragment (the 8 kg fragment) has a velocity of 2.5 m/s. As the centre of mass has a velocity of 10 m/s, the difference in velocities between the fragment and the centre of mass is 7.5 m/s.

$$v_{2\,rel} = 7.5\tfrac{m}{s}$$

(Don't bother about the sign of these velocities since v_{rel} will be squared.)

The kinetic energy is, therefore,

$$\begin{aligned} E_{k\,tot} &= \frac{1}{2}mv_{cm}^2 + \sum \frac{1}{2}m_i v_{i\,rel}{}^2 \\ &= \frac{1}{2}\cdot 20kg \cdot \left(10\tfrac{m}{s}\right)^2 + \left(\frac{1}{2}\cdot 12kg \cdot \left(5\tfrac{m}{s}\right)^2 + \frac{1}{2}\cdot 8kg \cdot \left(7.5\tfrac{m}{s}\right)^2\right) \\ &= \underbrace{1000J}_{\text{energy of the center of mass}} + \underbrace{375J}_{\text{energy relative to the center of mass}} \\ &= 1375J \end{aligned}$$

The result is the same for both methods. Here, the second method was longer, but it can become much shorter in some cases if some methods to simplify this calculation are used. This is what will be done for rotating objects in the next chapter.

SUMMARY OF EQUATIONS

Centre of Mass of a System Composed of Particles

$$\vec{r}_{cm} = \frac{1}{m}\sum \vec{r}_i m_i$$

In components:

$$x_{cm} = \frac{1}{m}\sum x_i m_i \qquad y_{cm} = \frac{1}{m}\sum y_i m_i \qquad z_{cm} = \frac{1}{m}\sum z_i m_i$$

Centre of Mass of a Rod

$$x_{cm} = \frac{1}{m}\int \lambda x\, dx$$

Total Momentum of a System

$$\vec{p}_{tot} = m\vec{v}_{cm}$$

Velocity of the Centre of Mass

$$\vec{v}_{cm} = \frac{1}{m}\sum m_i \vec{v}_i$$

In components:

$$v_{cm\,x} = \frac{1}{m}\sum m_i v_{xi} \qquad v_{cm\,y} = \frac{1}{m}\sum m_i v_{yi} \qquad v_{cm\,z} = \frac{1}{m}\sum m_i v_{zi}$$

Newton's First Law

$$m\vec{a}_{cm} = 0 \qquad \text{if} \qquad \sum \vec{F}_{ext} = 0$$

Newton's Second Law

$$\sum \vec{F}_{ext} = m\vec{a}_{cm}$$

or

$$\sum \vec{F}_{ext} = \frac{d\vec{p}_{tot}}{dt}$$

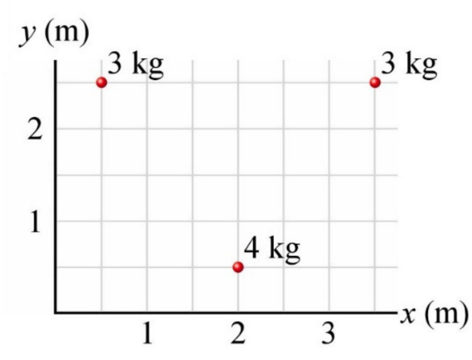
Total Kinetic Energy of a System

$$E_{k\,tot} = \frac{1}{2}mv_{cm}^2 + \sum \frac{1}{2}m_i v_{i\,rel}^2$$

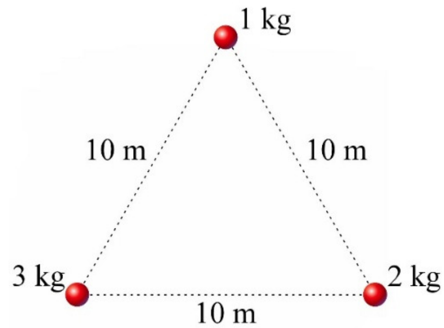
EXERCISES

11.1 Position of the Centre of Mass

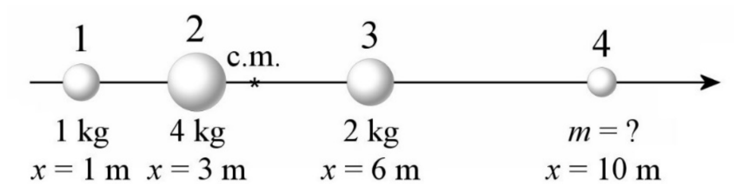
1. Where is the centre of mass of these three masses?



2. Where is the centre of mass of these three masses? (Put the origin of your axes at the position of the 3 kg mass.)

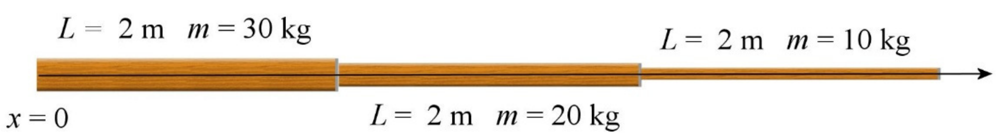


3. Knowing that the centre of mass of these 4 masses is at x = 4 m, find the mass of mass 4.

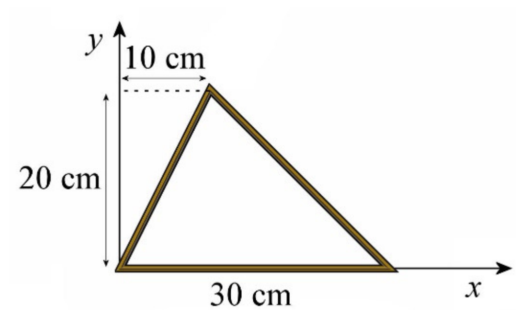


4. Where is the centre of mass of this 3 m long rod whose linear density is given by the formula $\lambda = 1\frac{kg}{m^3} \cdot x^2 + 1\frac{kg}{m}$?

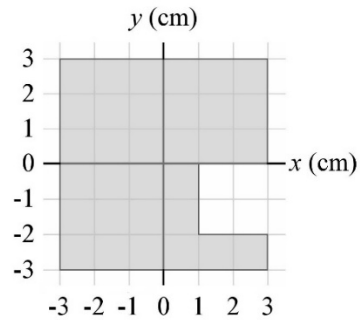
5. Where is the centre of mass of this stick composed of 3 rods?



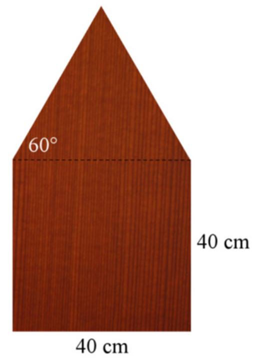
6. Where is the centre of mass of this object formed of three rods having the same linear density?



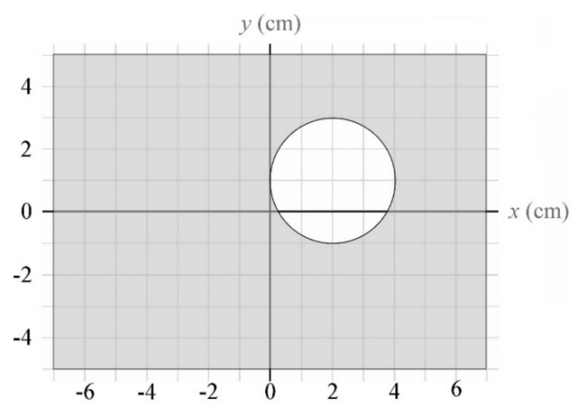
7. Where is the centre of mass of this plate having a uniform surface density?



8. Where is the centre of mass of this wooden plate having a density of 60 kg/m²?

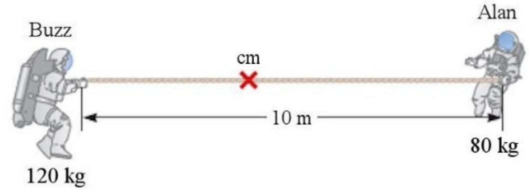


9. Where is the centre of mass of this plate having a uniform surface density if the hole has a radius of 2 cm?



11.2 Some Important Results Concerning the Centre of Mass

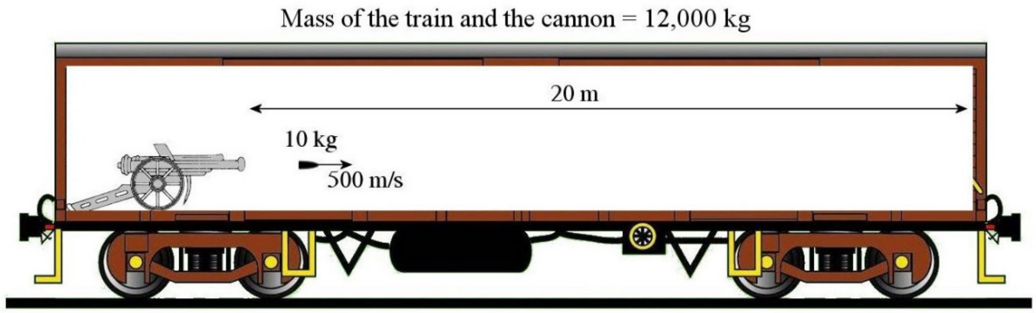
10. Buzz and Alan are two astronauts initially at rest floating in space as shown in the diagram. Buzz then pulls on the rope, which gives him a speed of 3 m/s directly towards Alan.



www.chegg.com/homework-help/questions-and-answers/astronauts-having-mass-m-connected-rope-length-d-having-negligible-mass-isolated-space-orb-q3272102

a) What is the velocity of Alan after Buzz has pulled on the rope?
b) What is the distance between Buzz and Alan 1 second after Buzz has pulled on the rope?
c) Where will they meet and after how much time?

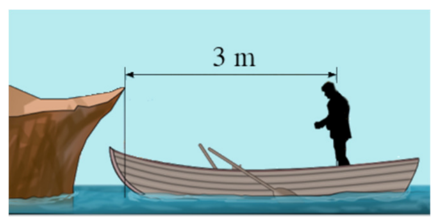
11. A train car containing a canon is at rest on a track. Suddenly, the gun accidentally shoots a 10 kg shell at 500 m/s. The canon is securely attached to the train, and there is no friction opposing the motion of the train.



www.sa-transport.co.za/train_drawings/wagons/fb-8.html
www.ipaustralia.com.au/applicant/the-arsenal-football-club-public-limited-company/trademarks/805631/

a) By how much distance did the train moved between the instant the gun was fired and the instant the shell reaches the other side of the train?
b) What is the velocity of the train after that the canon has fired the shell?
c) What is the speed of the centre of mass of the system formed by the train, the canon and the shells after that the canon has fired the shell?

12. Sebastien walks towards the front of his boat to go to the shore of the lake which is 3 m from him. When he arrives at the other end of the boat, the boat has moved 40 cm away from the shore. What is the mass of the boat if Sebastian has a mass of 60 kg?

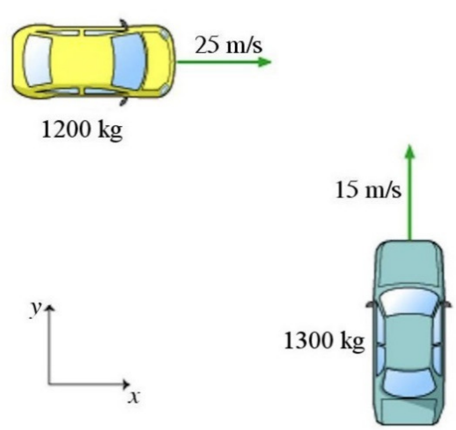


www.chegg.com/homework-help/questions-and-answers/mysterious-man-weight-wman-200-lb-boat-weight-wboat-400-lb-rest-close-river-edge-shown-pic-q947329

13. Here are two cars involved in a collision.

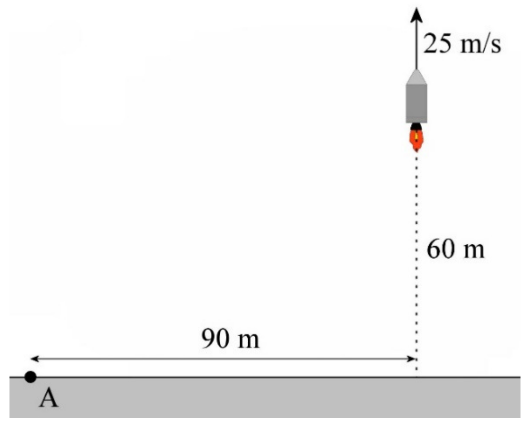
a) What is the velocity of the centre of mass before the collision?
b) What is the velocity of the centre of mass after the collision if the cars make a completely inelastic collision?
c) What is the velocity of the centre of mass after the collision if the cars make an elastic collision?

(Always give your answers in the form $\vec{v}_{cm} = \left(v_x\vec{i} + v_y\vec{j}\right)\frac{m}{s}$)

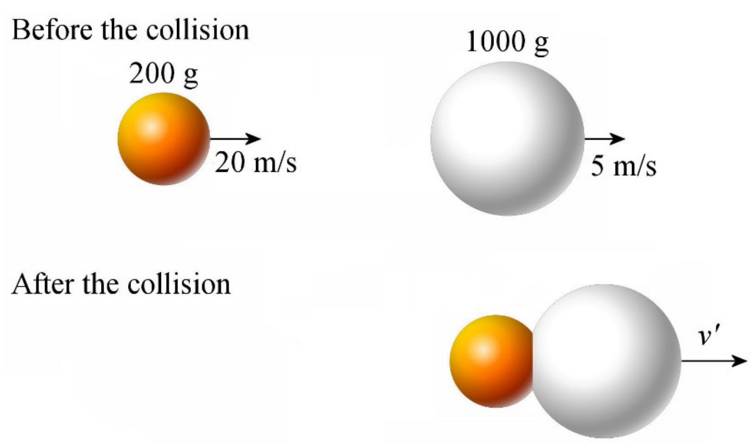


www.chegg.com/homework-help/questions-and-answers/problem-consider-collision-cars-initially-moving-right-angles-assume-collision-cars-stick--q3004672

14. A model rocket takes off vertically. When it reached an altitude of 60 m and a speed of 25 m/s, the engine stops, and the rocket explodes into two pieces. The first piece to reach the ground is a 500 g piece which touches the ground 6 seconds after the explosion at 90 m east of the point of departure of the rocket (point A in the diagram). Where is the other piece (whose mass is 1200 g) at that instant?



15. Here's a completely inelastic collision.



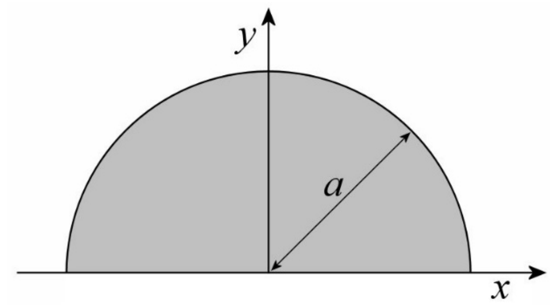
a) What is the velocity of the centre of mass before the collision?
b) What is the kinetic energy of the centre of mass before the collision?
c) What is kinetic energy relative to the centre of mass before the collision?
d) What is the speed *v*' after the collision?
e) What is the kinetic energy of the centre of mass after the collision?
f) What is kinetic energy relative to the centre of mass after the collision?

Challenges

(Questions more difficult than the exam questions.)

16. The density of a rod going from $x = 0$ to $x = L$ is given by $\lambda = kx^n$. What must be the value of n so that the centre of mass is at $x = 0.9 L$?

17. A plate having a uniform surface density has the shape of a semicircle. Where is the position of y-component of the centre of mass of this plate? (Give the position using the axes shown in the diagram.)



ANSWERS

11.1 Position of the Centre of Mass

1. $x_{cm} = 2$ m $y_{cm} = 1.7$ m
2. $x_{cm} = 4.17$ m $y_{cm} = 1.44$ m
3. 0.5 kg
4. $x_{cm} = 2.0625$ m
5. $x_{cm} = 2.333$ m
6. $x_{cm} = 13.98$ cm $y_{cm} = 6.28$ cm
7. $x_{cm} = -0.25$ cm $y_{cm} = 0.125$ cm
8. $x_{cm} = 20$ cm $y_{cm} = 29.53$ cm (origin at the lower left corner of the plate)
9. $x_{cm} = -0.19722$ m $y_{cm} = -0.0986$ m (origin at the centre of the rectangular plate)

11.2 Some Important Results Concerning the Centre of Mass

10. a) 4.5 m/s towards the left b) 2.5 m c) At the centre of mass, after 1.333 s
11. a) 1.665 cm towards the left b) 0.4167 m/s towards the left c) 0 m/s
12. 390 kg
13. a) $\vec{v}_{cm} = \left(12\vec{i} + 7{,}8\vec{j}\right)\frac{m}{s}$ b) $\vec{v}_{cm} = \left(12\vec{i} + 7{,}8\vec{j}\right)\frac{m}{s}$ c) $\vec{v}_{cm} = \left(12\vec{i} + 7{,}8\vec{j}\right)\frac{m}{s}$
14. The second piece is 37.5 m towards the east from the departure point, at an altitude of 47.6 m.
15. a) 7.5 m/s b) 33.75 J c) 18.75 J d) 7.5 m/s e) 33.75 J f) 0 J

Challenges

16. $n = 8$
17. $y_{cm} = 4a/3\pi$