# Newton's 2nd law in *Principia*

Here is how Newton stated his second law.

> *Law II: A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.*

No reference at all to mass and acceleration!

To understand what Newton means, we have to look some pages before in the Principia.

In reality, what is considered to be Newton's second law is much more similar to definition VIII. In this definition we have

> *Wherefore the accelerative force will stand in the same relation to the motive force, as speed does to quantity of motion. For the quantity of motion arises from the speed multiplied by the quantity of matter; and the motive force arises from the accelerative force multiplied by the same quantity of matter. For the sum of the actions of the accelerative force, upon the several particles of the body, is the motive force of the whole. Hence it is, that near the surface of the earth, where the accelerative gravity, or force productive of gravity, in all bodies is the same, the motive gravity or the weight is as the mass.*

The *accelerating force* is now called acceleration and *the motive force* is now called the force. So, he is clearly saying that $F = ma$. This shows that you must be quite careful when reading Newton's text (and all the texts of this era) because there are really a lot of different concepts that all bear the name of *force.*

He therefore defines force as the product *ma*. This definition is Newton's 2nd law in use today.

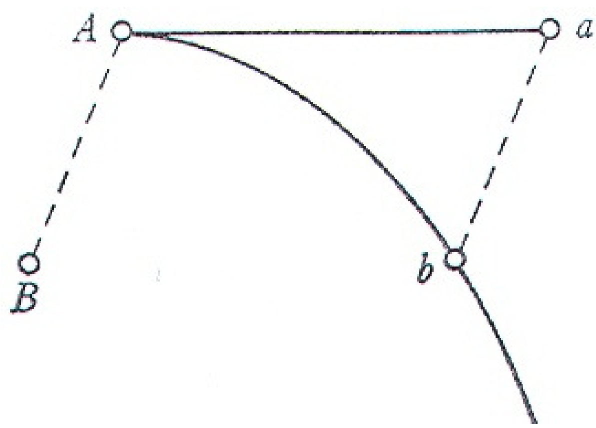
Newton's problem is how to use this definition to calculate the trajectory of planets, for example. Newton does not use algebraic methods as we do now, but geometric methods in which it is relatively difficult to represent acceleration. The 1st and 2nd laws will therefore specify how to calculate the motion of an object if the force is *ma.*

Newton separates the motion into two parts.

The object is initially at point *A* and it has a certain speed.

The first part of the motion (line *Aa*) is the displacement that the object would have if there were no force. The object would continue in a straight line at constant speed.

The second part is the displacement that the object would have if it were at rest at point *A* and a force were acting on it (line *AB*).




plato.stanford.edu/entries/newton-principia/#ThrEdiPri

When these two displacements are added as vectors, we arrive at the new position of the object (*b*).

The first law refers to the first part of this motion.

> *Law I: Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.*

The object continues its motion in a straight line at constant speed following the trajectory *Aa*. In the 3 preliminary versions of Principia made in 1684 and 1685, Newton still spoke of an impregnated force (*vis insista* in definition III). He gets rid of this force in this formulation, but it is still found everywhere in Principia.

The second law refers to the second part of this motion.

> *Law II: A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.*

The displacement *AB* is therefore proportional to *F*, therefore to *ma*.

Before interpreting all this, it should be known that when Newton speaks of *motion*, he often speaks of *quantity of motion* that he defined as *mv* (definition II). Some will say that if he had meant *quantity of motion,* he would have said *quantity of motion*. He could not have forgotten the term since the definition is only a few pages earlier in Principia. However, it is clear that Newton very often uses *motion* for *quantity of motion*. This excerpt from the explanation following the 3rd law shows this clearly.

> *Law III: To every action there is always opposed an equal reaction; or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.*

> *Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also pressed by the stone. If a horse draws a stone tied to a rope, the horse (if I may so say) will be equally drawn back towards the stone: for the distended rope, by the same endeavour to relax or unbend itself, will draw the horse as much towards the stone as it does the stone towards the horse, and will obstruct the progress of the one as much as it advances that of the other.*
>
> *If a body impinges upon another, and by its force change the motion of the other, that body also (became of the quality of, the mutual pressure) will undergo an equal change, in its own motion, towards the contrary part.*
>
> *The changes made by these actions are equal, not in the velocities but in the motions of bodies; that is to say, if the bodies are not hindered by any other impediments. For, because the motions are equally changed, the changes of the velocities made towards contrary parts are reciprocally proportional to the masses.*

Clearly, *motion* cannot be anything other than *quantity of motion* here.

The method is in fact very similar to the method used by Huygens to find the tension of a rope that holds an object that makes a circular motion (published in 1673). Newton says the method is not new.

Let's first see why the method works. We want to find the position *b* after a time $\Delta t$ with a constant force.

The first trajectory (*Aa*) is the one that the object would have had if there had been no force. The object would then follow a constant-speed motion in a straight line, and the displacement would be

$$\Delta\vec{r}_1 = \vec{v}_0 \Delta t$$

The second trajectory (*AB*) is the one that the object would have had if the object, initially at rest, had been subjected to a force. This displacement would be

$$\Delta\vec{r}_2 = \tfrac{1}{2}\vec{a}\left(\Delta t\right)^2$$

When these two motions are added, the total displacement is

$$\Delta\vec{r} = \vec{v}_0 \Delta t + \tfrac{1}{2}\vec{a}\left(\Delta t\right)^2$$

which makes it possible to know the final position of the object (*b*).

Now let's see how we can interpret what Newton says.

The whole thing can be seen as a simple multiplication by *m* of what was just done.

Multiplying by $m$, the length of the line $Aa$ then becomes

$$mv_0 \Delta t$$

Multiplying by $m$, the length of the line $AB$ then becomes

$$\tfrac{1}{2} ma(\Delta t)^2$$

Newton makes it clear that the length of this line is proportional to the motive force ($ma$) in his 2nd law and to the square of time in lemma X.

But since Newton works with proportions, the ratio of the lengths $Aa$ and $AB$ does not change when they are multiplied by the constant $m$. So, the result is the same. (Newton almost always works with proportional values as was the practice at the time.)

Others will argue that the force is, according to Newton's 2nd law,

$$F \propto \Delta(mv)$$

since he says that the change of *motion* is proportional to the force. As *motion* means *quantity of motion*, it would mean that the force is $\Delta(mv)$. They maintain that if Newton had meant $ma$, he would have said *rate of change of motion* rather than *change of motion*. Since the precision of language is not the strong point of the Principia, it is quite likely that Newton actually meant *rate of change* when he said *change*. However, it is true that Newton clearly uses $F \propto \Delta(mv)$ on several occasions even if this conflicts with definition VIII. For example, Newton uses it when he wants to find the shape of a planet's trajectory around the Sun. He then calculates the position of the planet at regular intervals $\Delta t$ and then makes this time $\Delta t$ tend towards 0 since the force is constantly changing.

In fact, everything works anyway because it's like multiplying our two vectors $\overrightarrow{\Delta r_1}$ and $\overrightarrow{\Delta r_2}$ by $m\Delta t$ (which is a constant when the calculation is done at regular intervals).

The length $AB$ then becomes

$$m\Delta t \tfrac{1}{2} a(\Delta t)^2 = m\Delta t \tfrac{1}{2} \frac{\Delta v}{\Delta t}(\Delta t)^2 = \tfrac{1}{2} m\Delta v(\Delta t)^2$$

Then $AB$ becomes proportional to the assumed motive force $\Delta(mv)$ and the square of time as specified by Newton.

But since Newton works with proportions, the ratio of the lengths $Aa$ and $AB$ does not change when they are multiplied by the constant $m\Delta t$. The method therefore still works even if the force is calculated with $\Delta(mv)$, as long as the time intervals are always the same. Note that Newton returns to $F = ma$ when the intervals are not constant (as in his analysis of motion in a resistant medium).

All this to say that the link between force and acceleration as well as the link between force and mass are far from obvious in Newton's *Principia*.