
INTRODUCTION

DATA ANALYSIS

MEASUREMENT AND UNCERTAINTY

Measurements are never infinitely accurate; an uncertainty on the measurement must be assessed for each measurement. Its value depends on the particular situation. The uncertainty of a measurement x is denoted Δx . The measurement is then written as follows:

$$x \pm \Delta x$$

This means that the exact value of the measurement is somewhere between $x - \Delta x$ and $x + \Delta x$. For example, if a measurement is 4.02 ± 0.34 , this means that the true value is surely somewhere between 3.68 and 4.36.

This uncertainty can come from two sources. These are:

1. Measuring device: Measuring devices do not have infinite precision.
2. Other factors:
 - a) Objects or quantities to be measured may be irregular, ill-defined or vary over time.
 - b) The way a device is used. The conditions in which the device is used may prevent us from taking advantage of the full precision of the latter.

Therefore, the uncertainty is

$$\Delta x = \Delta x_{\text{device}} + \Delta x_{\text{others}}$$

Evaluation of the Uncertainty due to the Measuring Device

- a) The uncertainty is given by the manufacturer of the device.

In this case, this uncertainty will be stated in the lab notebook. There are three

ways for the manufacturer to indicate the uncertainty.

1. The uncertainty is simply given.

This is the easiest case since the value of uncertainty is directly given. For example, the scales used in physics have an uncertainty of 0.1 g.

2. The relative uncertainty is given (percentage).

For example, if the measurement is 35.78 g and the uncertainty of the device is 3%, then the uncertainty is $0.03 \cdot 35.78 \text{ g} = 1.073 \text{ g}$.

3. The uncertainty on the last figure displayed by the device is given.

This can only happen with devices having a digital display. In this case, the manufacturer indicates the uncertainty on the last digit displayed by the unit. However, the wording is highly abbreviated. Thus, if there is an uncertainty of 2 on the last displayed number, the uncertainty is simply given as “ ± 2 digits”. Here are some examples of the values of the uncertainty in this case.

6.654 ± 1 digit	gives	6.654 ± 0.001
13.45 ± 3 digits	gives	13.45 ± 0.03
2.8324 ± 7 digits	gives	2.8324 ± 0.0007
63.67 ± 10 digits	gives	63.67 ± 0.10

Sometimes, the uncertainty is the sum of many of these cases. For example, let's suppose that the measurement on an ammeter is 100.6 mA. If the manufacturer of the ammeter indicates that the uncertainty of the device is $\pm 0.3\% + 1 \text{ mA} + 2$ digits, then the uncertainty is the sum of the three following elements:

0.3% uncertainty:	0.3 mA
1 mA uncertainty:	1 mA
2 digits uncertainty:	<u>0.2 mA</u>
Total uncertainty:	1.5 mA

- b) The uncertainty is not given by the manufacturer of the device.

- 1- **Without a digital display:** It is then assumed that the uncertainty due to the instrument is equal to half of the smallest graduation of the unit. The uncertainty of a ruler graduated at each millimetre is, therefore, 0.5 mm.

- 2- **With a digital display:** It is then assumed that the uncertainty is ± 1 on the last displayed digit.

Evaluation of the Uncertainty Due to Other Factors

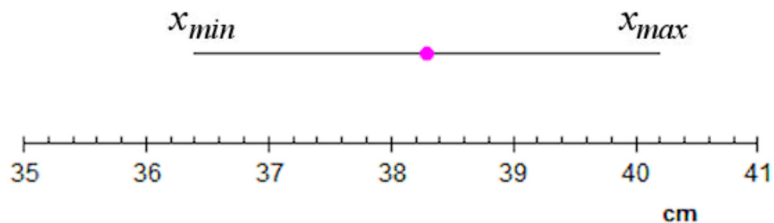
The value of this uncertainty depends on the situation. For example, in an optics laboratory, the position of the image of an object formed by a converging lens will be measured. To find the position of the image, the position where the image on a screen is sharp must be found. However, you'll soon realize that the image remains sharp even if you move the screen a little. The image begins to be sharp at a certain position ($x_{\min}$) and ceases to be sharp at another position ($x_{\max}$). The uncertainty, in this case, does not only come from the measuring device but also from the fact that there is a range of values for which the image is sharp. The measurement of time with a stopwatch is another example. As reflexes are not instantaneous, there is an additional uncertainty to the time measured. This additional uncertainty is approximately 0.2 s, which is much greater than the uncertainty of the stopwatch (0.01 s).

To properly evaluate the magnitude of the uncertainty in these cases, you must ask yourselves the following question: what are the extreme values that can represent with sufficient certainty the magnitude of the measured quantity?

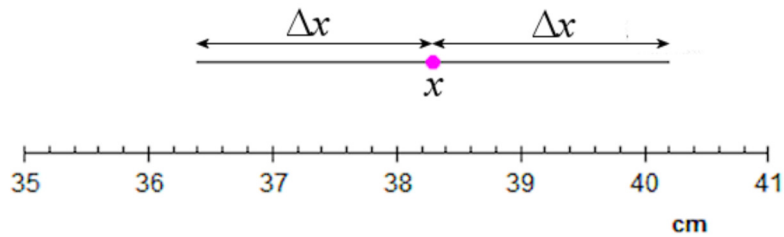
The largest possible value is noted $x_{\max}$. Here is an example to illustrate. Imagine that you measure the position of an image made by a lens. Moving the screen away from the lens, you estimate that the image ceases to be sharp when the screen is 40.2 cm from the lens. This is the maximum value of the position of the image. You are absolutely sure that the value cannot be larger than this value.

The smallest possible value is noted $x_{\min}$. Using our example, you now move the screen towards the lens to find the minimum distance. You then estimate that the image ceases to be sharp when the screen is 36.4 cm from the lens. This is the minimum value of the position of the image. You are absolutely sure that the value cannot be smaller than this value.

Now, you know that x is between $x_{\max}$ and $x_{\min}$. For the example of the image made by a lens, you know that the image is between 36.4 cm to 40.2 cm from the lens. Thus, a range of value is obtained.



This range of value can be represented with x and Δx .



As the x is midway between $x_{\max}$ and $x_{\min}$, it is the average of two values.

$$x = \frac{x_{\max} + x_{\min}}{2}$$

The uncertainty is equal to half of the gap between $x_{\max}$ and $x_{\min}$. Thus, it is

$$\Delta x_{\text{others}} = \frac{x_{\max} - x_{\min}}{2}$$

Don't forget that this uncertainty is only the uncertainty due to other factors. The uncertainty to the measuring device must be added to obtain the total uncertainty.

Example: During the measurement of a length L with a ruler (uncertainty of 1 mm), it is believed that the value of this length is certainly not larger than 12.8 cm and is certainly not smaller than 12.4 cm. Then

$$\begin{aligned} L &= \frac{12.8\text{cm} + 12.4\text{cm}}{2} \\ &= 12.6\text{cm} \end{aligned}$$

and

$$\begin{aligned} \Delta L &= \frac{12.8\text{cm} - 12.4\text{cm}}{2} \\ &= 0.2\text{cm} \end{aligned}$$

Thus, the total uncertainty is

$$\begin{aligned} \Delta L &= \Delta L_{\text{device}} + \Delta L_{\text{others}} \\ &= 0.1\text{cm} + 0.2\text{cm} \\ &= 0.3\text{cm} \end{aligned}$$

Therefore, the result is

$$L = 12.6 \pm 0.3 \text{ cm}$$

This means that the maximum possible value of the measurement is $12.6 \text{ cm} + 0.3 \text{ cm} = 12.9 \text{ cm}$ and that the minimum possible value of the

length is $12.6 \text{ cm} - 0.3 \text{ cm} = 12.3 \text{ cm}$.

How to Correctly Write a Measurement and Its Uncertainty

The measure and its uncertainty must be written correctly. In order to do that, these rules must be followed:

1) The uncertainty must have only one or two significant digits.

Perhaps you are unfamiliar with the notion of significant digits, or maybe it is lost in your childhood memories. So here is a quick review of what a significant digit is.

- a) All digits different from 0 are significant.
- b) All 0 located between digits different from 0 are significant.
- c) All 0 following a digit different from 0 are significant.
- d) All 0 which are not preceded by a digit different from 0 are not significant because they only serve to locate the decimal point.
- e) One or more consecutive 0 placed at the end of a whole number are generally not significant.

2) The measurement must have the same precision as the uncertainty.

If, according to the previous rule, the uncertainty was rounded to the nearest hundredth, the measurement must also be rounded to the nearest hundredth. If the uncertainty was rounded to the nearest thousandth, the measurement must also be rounded to the nearest thousandth. Thus, the measurement or the result of a calculation cannot be rounded before the uncertainty is known because the precision is determined by the uncertainty.

Here are some examples of the application of these rules.

Calculations gave $T = 18.4278$ and $\Delta T = 1.654$.

First, we must take care of the uncertainty which has 4 significant digits. As only 2 significant digits must remain, the uncertainty is rounded to $\Delta T = 1.7$. T must then be rounded to the same precision as the uncertainty. As the uncertainty was rounded to the nearest tenth to get two digits significant, the value of T must also be rounded to the nearest tenth to obtain $T = 18.4$. The answer is, therefore, $T = 18.4 \pm 1.7$.

Calculations gave $p = 0.084056$ and $\Delta p = 0.000603393$.

As always, the uncertainty is dealt with first. To reduce the number of significant digits

to 2 for the uncertainty, it must be rounded to $\Delta p = 0.00060$. Note that the 0 after the 6 was kept because it is a significant digit (rule c). Then, as the uncertainty has been rounded to the nearest hundred thousandths (5th digit after the decimal point), the value from p must be rounded to the nearest hundred thousandths to obtain $p = 0.08406$. The answer is thus $p = 0.08406 \pm 0.00060$ or $p = (840.6 \pm 6.0) \times 10^{-4}$.

Calculations gave $\nu = 285,759,546$ and $\Delta\nu = 67,749$.

If the uncertainty is rounded to have two significant figures, $\Delta\nu = 68,000$ is obtained. Note that the three 0 in 68,000 are not significant (rule e). As the uncertainty has been rounded to the nearest thousands, the value of ν must also be rounded to the nearest thousands to get $\nu = 285,760,000$. The final value can then be written as $\nu = 285,760,000 \pm 68,000$ or $\nu = (28,576.0 \pm 6.8) \times 10^4$.

These examples show that the uncertainty has only two significant digits while the measurement may have more. In general, the experiment is considered to be more accurate if the measurement has a lot of significant digits. Thus, in the examples, the measurement of T is not very accurate since T has only 3 significant digits. On the other hand, the measurement of ν is much more accurate since ν still has 6 significant digits. It will be difficult to obtain high-precision results in the experiments performed in the college but it can be achieved in major scientific laboratories (are you surprised?). At the moment, one of the most accurate scientific experiments is the measurement of the anomalous magnetic moment of the electron. The measured value is

$$(11,596,521,807.6 \pm 2.7) \times 10^{-13}$$

This experiment is remarkably accurate since the measurement has 12 significant digits (which, by the way, all agree with the theoretical predictions of quantum electrodynamics).

Here is another example that will illustrate the subtlety of the rules: assume that a calculation gave $R = 0.086$ and $\Delta R = 3.462$.

If the uncertainty is rounded to keep only two significant digits, $\Delta R = 3.5$ is obtained. As it was rounded to the nearest tenth, R must also be rounded to the nearest tenth to get $R = 0.1$. The result is then $R = 0.1 \pm 3.5$. It may seem surprising to have a value of R smaller than its uncertainty, but this can sometimes happen. This will generally happen if the theoretical value of R is 0.

Finally, what happens if the result of a calculation is $F = 18.4$ and $\Delta F = 0.0063$?

In this case, the uncertainty already has 2 significant digits. As it is rounded to the nearest ten thousandth, F must also be rounded to the nearest ten thousandth. However, this is impossible to do since the value of F stops at the tenth. In this case,

the error probably comes from the fact that the value of F was rounded right after calculating it and it was then decided to round it to the nearest tenth. To remedy this situation, the value of F must be recalculated to know the missing digits in order to be able to correctly round it to the nearest ten thousandth. If the calculation is redone and you realize that the value is exactly 18.4, then this means that all the other digits of F are zeros. In this case, the value of F is $F = 18.4000 \pm 0.0063$.

EXERCISES

Write those measurements correctly

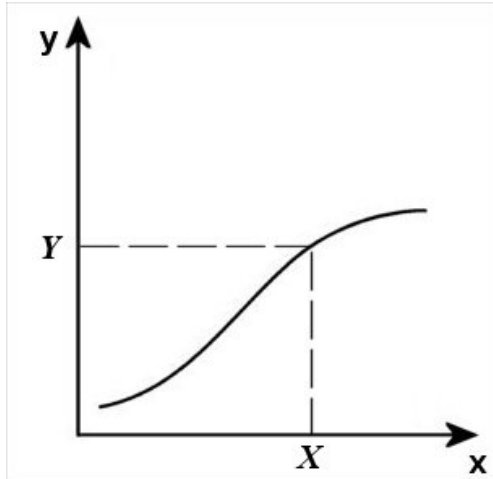
- | | |
|---|---|
| 1 - 96.6745 ± 3.7006 | $\underline{\hspace{1cm} \pm \hspace{1cm}}$ |
| 2 - 135.8 ± 5.943 | $\underline{\hspace{1cm} \pm \hspace{1cm}}$ |
| 3 - 0.00067 ± 0.022 | $\underline{\hspace{1cm} \pm \hspace{1cm}}$ |
| 4 - $295,450 \pm 22,677$ | $\underline{\hspace{1cm} \pm \hspace{1cm}}$ |
| 5 - 5603 ± 113 | $\underline{\hspace{1cm} \pm \hspace{1cm}}$ |
| 6 - $8.9875 \times 10^{-2} \pm 0.00045$ | $\underline{\hspace{1cm} \pm \hspace{1cm}}$ |

For each of these measurements, specify the magnitude of the measurement and its uncertainty

- | | |
|---|---|
| 1 - The width of this page | $\underline{\hspace{1cm} \pm \hspace{1cm}}$ |
| 2 - The position of the image | $\underline{\hspace{1cm} \pm \hspace{1cm}}$ |
| 3 - The mass | $\underline{\hspace{1cm} \pm \hspace{1cm}}$ |
| 4 - The period of oscillation of the mass | $\underline{\hspace{1cm} \pm \hspace{1cm}}$ |
| 5 - The room temperature | $\underline{\hspace{1cm} \pm \hspace{1cm}}$ |
| 6 - The current in the circuit | $\underline{\hspace{1cm} \pm \hspace{1cm}}$ |
| 7 - The resistance | $\underline{\hspace{1cm} \pm \hspace{1cm}}$ |

UNCERTAINTY CALCULATIONS WITH THE METHOD OF EXTREME VALUES

Certain quantities cannot be directly measured. For example, the average speed between two points of an object in a rectilinear motion cannot be directly measured. To find it, the distance between these two points must be measured, and then the time it takes for the object to cover that distance is measured. The average speed is simply the ratio of these two values. Then, what is the uncertainty of this calculated speed?

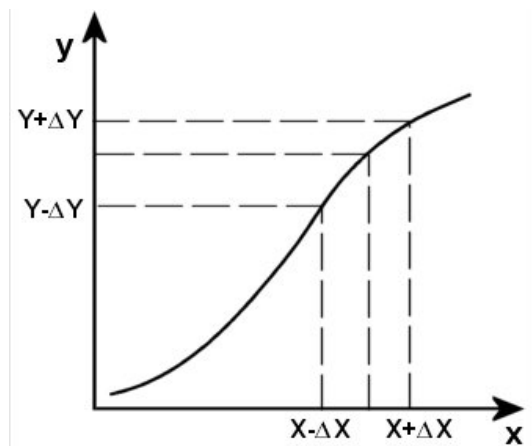


Suppose that y must be calculated from x with a function (like $y = x^2$). This means that

$$y = f(x)$$

Suppose that the graph of this function is shown in the figure. If the variable x has a certain value (written as X), the function gives the value of the variable y (written as Y).

If ΔX is the uncertainty of X , the uncertainty of Y can be determined by evaluating the extreme values of Y from the extreme values of X . The second figure shows what this means. The value of the function is calculated for $X + \Delta X$. The maximum value of Y , which is $Y + \Delta Y$, is then obtained. The value of the function for $X - \Delta X$ is then calculated to obtain the minimum value of Y , which is $Y - \Delta Y$. These two calculations give the extreme values of Y . The maximum value of X does not necessarily correspond to the maximum value of Y . If the slope is negative, the maximum value of X corresponds instead to the minimum value of Y .



From the extreme values of Y , the value of Y and its uncertainty can be calculated. This is the *method of extreme values*.

Let's take an example to illustrate this method. Suppose that a stone is dropped from the top of a cliff, and it took 1.5 ± 0.1 s for the stone to reach the bottom of the cliff. The height of the cliff can then be found with the following formula.

$$h = \frac{1}{2} g t^2$$

If the fall lasted 1.5 s, then the height is

$$h = \frac{1}{2} \cdot 9.8 \frac{m}{s^2} \cdot (1.5s)^2 = 11.025m$$

However, as there is an uncertainty on the time, then there must be an uncertainty on the height of the cliff obtained from this calculation. That extreme values of the height can be found from the extreme values of the time. As the maximum time is 1.6 sec (1.5 s + 0.1 s), the maximum height is

$$h = \frac{1}{2} \cdot 9.8 \frac{m}{s^2} \cdot (1.6s)^2 = 12.544m$$

As the minimum time is 1.4 s (1.5 s – 0.1s), the minimum height is

$$h = \frac{1}{2} \cdot 9.8 \frac{m}{s^2} \cdot (1.4s)^2 = 9.604m$$

The height of the cliff is therefore between 9.604 m and 12.544 m. The value of height is, therefore,

$$h = \frac{12.544m + 9.604m}{2} = 11.074m$$

and its uncertainty is

$$\Delta h = \frac{12.544m - 9.604m}{2} = 1.47m$$

This gives $h = 11.07 \text{ m} \pm 1.47 \text{ m}$. If it is written with the correct number of significant digits, the final result is $h = 11.1 \text{ m} \pm 1.5 \text{ m}$.

A few simple rules for uncertainty calculation

Using the method of extreme values, some simple rules will be obtained for the addition, subtraction, and multiplication.

Addition $z = x + y$

The maximum value of an addition is obtained by adding up the maximum values of each variable.

$$\begin{aligned}
 z_{\max} &= x_{\max} + y_{\max} \\
 &= (x + \Delta x) + (y + \Delta y)
 \end{aligned}$$

The minimum value of an addition is obtained by summing the minimum values of each variable.

$$\begin{aligned}
 z_{\min} &= x_{\min} + y_{\min} \\
 &= (x - \Delta x) + (y - \Delta y)
 \end{aligned}$$

Thus, the uncertainty is

$$\begin{aligned}
 \Delta z &= \frac{z_{\max} - z_{\min}}{2} \\
 &= \frac{[(x + \Delta x) + (y + \Delta y)] - [(x - \Delta x) + (y - \Delta y)]}{2}
 \end{aligned}$$

Simplifying, the result is

Uncertainty of the addition $z = x + y$

$$\Delta z = \Delta x + \Delta y$$

Subtraction $z = x - y$

The maximum value of the subtraction $x - y$ is obtained by subtracting the minimum value of y to the maximum value of x .

$$\begin{aligned}
 z_{\max} &= x_{\max} - y_{\min} \\
 &= (x + \Delta x) - (y - \Delta y)
 \end{aligned}$$

The minimum value of the subtraction $x - y$ is obtained by subtracting the maximum value of y to the minimum value of x .

$$\begin{aligned}
 z_{\min} &= x_{\min} - y_{\max} \\
 &= (x - \Delta x) - (y + \Delta y)
 \end{aligned}$$

Thus, the uncertainty is

$$\begin{aligned}
 \Delta z &= \frac{z_{\max} - z_{\min}}{2} \\
 &= \frac{[(x + \Delta x) - (y - \Delta y)] - [(x - \Delta x) - (y + \Delta y)]}{2}
 \end{aligned}$$

Simplifying, the result is

Uncertainty of the subtraction $z = x - y$

$$\Delta z = \Delta x + \Delta y$$

Multiplication $z = xy$

The maximum value of a multiplication is obtained by multiplying the maximum values of each variable.

$$\begin{aligned} z_{\max} &= x_{\max} y_{\max} \\ &= (x + \Delta x)(y + \Delta y) \end{aligned}$$

The minimum value of a multiplication is obtained by multiplying the minimum values of each variable.

$$\begin{aligned} z_{\min} &= x_{\min} y_{\min} \\ &= (x - \Delta x)(y - \Delta y) \end{aligned}$$

Thus, the uncertainty is

$$\begin{aligned} \Delta z &= \frac{z_{\max} - z_{\min}}{2} \\ &= \frac{[(x + \Delta x)(y + \Delta y)] - [(x - \Delta x)(y - \Delta y)]}{2} \end{aligned}$$

Simplifying, the result is

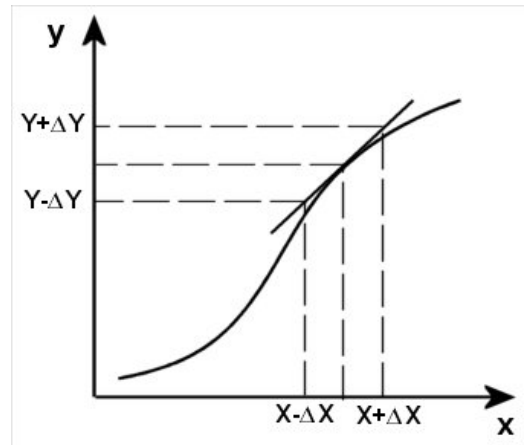
Uncertainty of the multiplication $z = x y$

$$\Delta z = x\Delta y + y\Delta x$$

We could go on like this to obtain other formulas. However, all these results can be obtained much more easily with calculus.

UNCERTAINTY CALCULATION WITH CALCULUS

Here, an approximation of the method of extreme values will be used to calculate the uncertainty. For this approximation, a tangent line to the function at point X is used to calculate the extreme values instead of the function. The figure on the right shows this. If the uncertainties are small, the values obtained from the tangent line are almost identical to the values obtained from the function. But be warned; if the uncertainties are too large or if the function varies very quickly, the calculation of uncertainties with the tangent line will be not very accurate.



The slope at point X must be calculated first. Fortunately, a very powerful method can be used to calculate a slope: the derivative. With the function $y = f(x)$, the slope of the function is given by

$$\text{slope} = \frac{dy}{dx}$$

However, the slope of the tangent line is also given by

$$\text{slope} = \frac{\Delta y}{\Delta x}$$

As these two previous equations must be equal, we have

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

If this equation is solved for Δy , an approximation for the uncertainty is obtained.

$$\Delta y = \left| \frac{dy}{dx} \right| \Delta x$$

The absolute value of the derivative must be taken to have a positive uncertainty. A negative uncertainty would change the “plus or minus” sign in front of the uncertainty to a “minus or plus” sign, which doesn’t make any difference.

Let’s revisit our free-fall example to see if the approximation is good. We already know that the height of the cliff is 11.025 m if the height is calculated with the free-fall time (1.5 s). Now, the uncertainty of this result will be calculated with the approximation. Here

we do not have the variables x and y , but rather h and t . However, the principle is the same. To obtain the uncertainty of a variable (here h), just derive the function giving this variable (here $\frac{1}{2}gt^2$) and multiply it by the known uncertainty (here Δt). Therefore, the uncertainty is

$$\Delta h = \left| \frac{d\left(\frac{1}{2}gt^2\right)}{dt} \right| \Delta t$$

$$= |gt| \Delta t$$

It only remains to calculate the value of this uncertainty to obtain

$$\Delta h = \left| 9.8 \frac{m}{s^2} \cdot 1.5s \right| \cdot 0.1s$$

$$= 1.47m$$

This uncertainty is very similar to the one obtained by calculating the uncertainty with the method of extreme values (1.47 m).

The height of the cliff is therefore $11.025 \text{ m} \pm 1.47 \text{ m}$ according to this method. Written with the correct number of significant digits, the height is $11.0 \text{ m} \pm 1.5 \text{ m}$.

In summary, the calculation is first made without worrying about the uncertainty. Then, the uncertainty is calculated with the derivative.

Examples Uncertainty Calculation for Some Functions

a) $y = x^4$ $x = 3.0 \pm 0.1$

$$y = 3^4 = 81$$

$$\Delta y = 4 x^3 \Delta x = 4 \cdot 3^3 \cdot 0.1 = 10.8$$

$$\text{Therefore, } y = 81 \pm 11$$

b) $R = e^t$ $t = 3.6 \pm 0.1$

$$R = e^{3.6} = 36.598$$

$$\Delta R = e^t \Delta t = e^{3.6} \cdot 0.1 = 3.66$$

$$\text{Therefore, } R = 36.6 \pm 3.7$$

$$\text{c) } S = \ln N \quad N = 34 \pm 1$$

$$S = \ln 34 = 3.5264$$

$$\Delta S = \frac{1}{N} \cdot \Delta N = \frac{1}{34} \cdot 1 = 0.02941$$

$$\text{Therefore, } S = 3.526 \pm 0.029$$

Note that in calculations involving trigonometric functions, the uncertainties of the angles must always be in radians and not in degrees. All the other angles can be in degrees but the uncertainty of the angle in calculations involving trigonometric function must be in radians.

Example

$$p = \sin w \quad w = 43.0 \pm 0.5^\circ$$

$$p = \sin 43^\circ = 0.68199$$

$$\Delta p = |\cos w| \Delta w = |\cos 43^\circ| \cdot 0.00873 \text{ rad} = 0.0063$$

$$\text{Therefore, } p = 0.6820 \pm 0.0063$$

If an angle is calculated from an inverse trigonometric function, the uncertainty obtained is in radians.

Example

$$\theta = \arctan r \quad r = 2.045 \pm 0.038$$

$$\theta = \arctan 2.045 = 63.9415^\circ$$

$$\Delta \theta = \left| \frac{1}{1+r^2} \right| \Delta r = \left| \frac{1}{1+2.045^2} \right| \cdot 0.038 = 0.00733 \text{ rad}$$

This value can be changed in degrees to obtain 0.42°

$$\text{Therefore, } \theta = 63.94^\circ \pm 0.42^\circ$$

EXERCISES

$$\text{a) Calculate} \quad y = 5x^3 \quad x = 2.23 \pm 0.06$$

$$\text{b) Calculate} \quad s = 4 \cos v \quad v = 35.3^\circ \pm 0.5^\circ$$

Answer

a) $y = 55.4 \pm 4.5$

b) $s = 3.264 \pm 0.020$

UNCERTAINTY CALCULATIONS AND MULTIVARIABLE CALCULUS

Now suppose that z must be calculated from two measured values x and y . This means that

$$z = f(x, y)$$

It can be shown that the uncertainty is then given by the following equation.

$$\Delta z = \left| \frac{\partial z}{\partial x} \right| \Delta x + \left| \frac{\partial z}{\partial y} \right| \Delta y$$

where the derivatives are partial derivatives (see box).

Partial Derivatives

The partial derivative $\frac{\partial f}{\partial x}$ is the derivative of f with respect to x only. This means that y is considered as being a constant in the function. For example, if $z = x^2 y^3 + y^2$, then

$$\frac{\partial f}{\partial x} = 2xy^3$$

The partial derivative $\frac{\partial f}{\partial y}$ is the derivative of f with respect to y only. This means that x is considered as being a constant in the function. For example, if $z = x^2 y^3 + y^2$, then

$$\frac{\partial f}{\partial y} = 3x^2 y^2 + 2y$$

To illustrate how to use this formula, let's revisit our example of the stone thrown from the top of a cliff by adding an uncertainty on g .

$$g = 9.81 \text{ m/s}^2 \pm 0.01 \text{ m/s}^2$$

The height of the cliff is then

$$h = \frac{1}{2} g t^2 = \frac{1}{2} \cdot 9.81 \frac{m}{s^2} \cdot (1.5s)^2 = 11.03625m$$

The uncertainty of the height of the cliff will be larger than before since the two variables have uncertainties now. To obtain the uncertainty, the formula of the height must be derived with respect to each variable having an uncertainty. The total uncertainty of the height is then the sum of the derivatives multiplied by the uncertainties. Here, the two variables are g and t and the uncertainty of h is, therefore,

$$\Delta h = \left| \frac{\partial h}{\partial t} \right| \Delta t + \left| \frac{\partial h}{\partial g} \right| \Delta g$$

The partial derivative of the function h with respect to t is

$$\left| \frac{\partial h}{\partial t} \right| = g t$$

and the partial derivative of the function h with respect to g is

$$\left| \frac{\partial h}{\partial g} \right| = \frac{1}{2} t^2$$

Therefore, the uncertainty of the height of the cliff is

$$\Delta h = g t \Delta t + \frac{1}{2} t^2 \Delta g$$

It only remains to put the numbers in this equation to obtain the uncertainty of h .

$$\begin{aligned} \Delta h &= 9.81 \frac{m}{s^2} \cdot 1.5s \cdot 0.1s + \frac{1}{2} \cdot (1.5s)^2 \cdot 0.01 \frac{m}{s^2} \\ &= 1.4715m + 0.01125m \\ &= 1.48275m \end{aligned}$$

The height is then

$$h = 11.0 \pm 1.5 \text{ m}$$

The uncertainty of the height has not increased much by adding an uncertainty of 0.01 m/s^2 to g . The calculation also indicated that uncertainty of the height due to the uncertainty of g is 0.01125 m while the uncertainty due to the uncertainty of t is 1.47 m . It is quite clear that if we want to improve this experiment and reduce the uncertainty of the height, nothing much is gained by reducing the uncertainty of g . It would be better to focus on ways to improve the time measurement.

Examples Uncertainty Calculation on Multivariable Functions

a) $z = \frac{x}{y}$ $x = 3.04 \pm 0.24$ $y = 5.78 \pm 0.11$

The function is

$$z = \frac{3.04}{5.78} = 0.525895$$

The uncertainty is given by

$$\Delta z = \left| \frac{\partial z}{\partial x} \right| \Delta x + \left| \frac{\partial z}{\partial y} \right| \Delta y$$

As

$$\frac{\partial z}{\partial x} = \frac{1}{y}$$

$$\frac{\partial z}{\partial y} = \frac{-x}{y^2}$$

the uncertainty is

$$\begin{aligned} \Delta z &= \left| \frac{1}{y} \right| \Delta x + \left| \frac{-x}{y^2} \right| \Delta y \\ &= \left| \frac{1}{5.78} \right| \cdot 0.24 + \left| \frac{-3.04}{5.78^2} \right| \cdot 0.11 \\ &= 0.0415 + 0.0100 \\ &= 0.0515 \end{aligned}$$

Here, the uncertainty of z originates mainly from the uncertainty of x since the first term is about 4 times larger than the second.

The final result is $z = 0.526 \pm 0.052$

b) $w = \frac{xy}{z}$ $x = 22.3 \pm 0.2$ $y = 4.6 \pm 0.1$ $z = 5.1 \pm 0.3$

The function is

$$w = \frac{22.3 \cdot 4.6}{5.1} - 5.1 = 15.0137$$

The previous ideas can easily be generalized to conclude that, with three variables, the uncertainty is

$$\Delta w = \left| \frac{\partial w}{\partial x} \right| \Delta x + \left| \frac{\partial w}{\partial y} \right| \Delta y + \left| \frac{\partial w}{\partial z} \right| \Delta z$$

As

$$\frac{\partial w}{\partial x} = \frac{y}{z}$$

$$\frac{\partial w}{\partial y} = \frac{x}{z}$$

$$\frac{\partial w}{\partial z} = \frac{-xy}{z^2} - 1$$

the uncertainty is

$$\begin{aligned} \Delta w &= \left| \frac{y}{z} \right| \Delta x + \left| \frac{x}{z} \right| \Delta y + \left| \frac{-xy}{z^2} - 1 \right| \Delta z \\ &= \left| \frac{4.6}{5.1} \right| \cdot 0.2 + \left| \frac{22.3}{5.1} \right| \cdot 0.1 + \left| \frac{-22.3 \cdot 4.6}{5.1^2} - 1 \right| \cdot 0.3 \\ &= 0.1804 + 0.4373 + 1.4832 \\ &= 2.1009 \end{aligned}$$

These results indicate that the uncertainty of w comes mainly from the uncertainty of z .

The final result is $w = 15.0 \pm 2.1$

$$\text{c) } I = \frac{x^3}{y^2 + x^2} + y \quad x = 3.04 \pm 0.07 \quad y = 5.46 \pm 0.05$$

The function is

$$I = \frac{3.04^3}{5.46^2 + 3.04^2} + 5.46 = 6.17939$$

The uncertainty is

$$\Delta I = \left| \frac{\partial I}{\partial x} \right| \Delta x + \left| \frac{\partial I}{\partial y} \right| \Delta y$$

The result is

$$\begin{aligned}
\Delta I &= \left| \frac{3x^2(x^2 + y^2) - x^3(2x)}{(x^2 + y^2)^2} \right| \Delta x + \left| -\frac{x^3}{(x^2 + y^2)^2} 2y + 1 \right| \Delta y \\
&= \left| \frac{3 \cdot 3.04^2 \cdot (3.04^2 + 5.46^2) - 3.04^3 \cdot 2 \cdot 3.04}{(3.04^2 + 5.46^2)^2} \right| \cdot 0.07 + \left| -\frac{3.04^3}{(3.04^2 + 5.46^2)^2} \cdot 2 \cdot 5.46 + 1 \right| \cdot 0.05 \\
&= 0.041853 + 0.03994 \\
&= 0.081793
\end{aligned}$$

Therefore, $I = 6,179 \pm 0,082$.

Here, the uncertainty comes almost equally from the uncertainty of each variable.

d) $\theta = \arcsin \frac{a}{b}$ $a = 0.345 \pm 0.012$ $b = 0.872 \pm 0.022$

The value of θ is

$$\theta = \arcsin \frac{a}{b} = \arcsin \frac{0.345}{0.872} = 23.30603^\circ$$

The uncertainty is

$$\Delta \theta = \left| \frac{\partial \theta}{\partial a} \right| \Delta a + \left| \frac{\partial \theta}{\partial b} \right| \Delta b$$

Which is

$$\Delta \theta = \left| \frac{1}{\sqrt{1 - \left(\frac{a}{b}\right)^2}} \frac{1}{b} \right| \Delta a + \left| \frac{1}{\sqrt{1 - \left(\frac{a}{b}\right)^2}} \frac{a}{b^2} \right| \Delta b$$

Therefore,

$$\begin{aligned}
\Delta \theta &= 0.014984 \text{ rad} + 0.010869 \text{ rad} \\
&= 0.025853 \text{ rad} \\
&= 1.48^\circ
\end{aligned}$$

The final result is then $\theta = 23.3^\circ \pm 1.5^\circ$.

Do not forget that the uncertainty of an angle is always in radians when the angle is calculated with an inverse trigonometric function.

The following example shows that the equation must not be necessarily solved for the variable for which the uncertainty is sought. The uncertainty can be calculated by deriving implicitly. For this to work, the variable for which the uncertainty is sought must be the only variable on one side of the equation and all the other variables must be on the other side of the equation.

$$\text{e) } \sin \theta = \frac{a}{b} \quad a = 0.345 \pm 0.012 \quad b = 0.872 \pm 0.022$$

Those are the same values as in the previous example.

The value of θ is

$$\theta = \arcsin \frac{a}{b} = \arcsin \frac{0.345}{0.872} = 23.30603^\circ$$

To obtain the uncertainty, each side of the equation is derived.

$$\left| \frac{\partial \sin \theta}{\partial \theta} \right| \Delta \theta = \left| \frac{\partial (a/b)}{\partial a} \right| \Delta a + \left| \frac{\partial (a/b)}{\partial b} \right| \Delta b$$

This gives

$$|\cos \theta| \Delta \theta = \left| \frac{1}{b} \right| \Delta a + \left| \frac{a}{b^2} \right| \Delta b$$

The uncertainty is thus

$$\begin{aligned} \Delta \theta &= \frac{\left| \frac{1}{b} \right| \Delta a + \left| \frac{a}{b^2} \right| \Delta b}{|\cos \theta|} \\ &= \frac{\frac{1}{0.872} \cdot 0.012 + \frac{0.345}{0.872^2} \cdot 0.022}{\cos 23.30603^\circ} \\ &= 0.025853 \text{ rad} \end{aligned}$$

The final result is, therefore, $\theta = 23.3^\circ \pm 1.5^\circ$

This method can greatly simplify some uncertainty calculations. For example, the derivative of an inverse sine function (which few remember) was avoided in this last example. The derivative of a sine function had to be made instead (for which the derivative is well known to all...).

A few simple rules for uncertainty calculation

With this method, it can be shown that the simple rules for the addition, subtraction and multiplication were correct.

Addition $z = x + y$

The uncertainty is

$$\begin{aligned}\Delta z &= \left| \frac{\partial z}{\partial x} \right| \Delta x + \left| \frac{\partial z}{\partial y} \right| \Delta y \\ &= |1 + 0| \Delta x + |0 + 1| \Delta y \\ &= \Delta x + \Delta y\end{aligned}$$

Subtraction $z = x - y$

The uncertainty is

$$\begin{aligned}\Delta z &= \left| \frac{\partial z}{\partial x} \right| \Delta x + \left| \frac{\partial z}{\partial y} \right| \Delta y \\ &= |1 + 0| \Delta x + |0 - 1| \Delta y \\ &= \Delta x + \Delta y\end{aligned}$$

Multiplication $z = xy$

The uncertainty is

$$\begin{aligned}\Delta z &= \left| \frac{\partial z}{\partial x} \right| \Delta x + \left| \frac{\partial z}{\partial y} \right| \Delta y \\ &= |y| \Delta x + |x| \Delta y \\ &= y \Delta x + x \Delta y\end{aligned}$$

EXERCISES

Calculate z and Δz if x and y are

$$x = 12.70 \pm 0.10$$

$$y = 6.30 \pm 0.20$$

a) $z = x + y$

b) $z = x - y$

c) $z = xy$

d) $z = \frac{x}{y}$

e) $z = 4xy$

f) $\frac{1}{z} = \frac{1}{x} + \frac{1}{y}$

g) $z = 2\pi e^{(xy/10)}$

h) $z = y \ln x$

$$\text{i) } z = \frac{x+y}{x}$$

$$\text{j) } e^z = \sqrt{\frac{x}{y}}$$

$$\text{k) } z = x^2y + x^3$$

$$\text{l) } \tan z = \frac{x}{y}$$

Calculate z and Δz if θ and A are

$$\theta = 34.6^\circ \pm 1.2^\circ$$

$$A = 23.47 \pm 0.53$$

$$\text{m) } z = A \cos \theta$$

$$\text{n) } z = A \sin^2 \theta + A \cos^2 \theta$$

Answers (with an uncertainty having 2 significant digits)

$$\text{a) } 19.00 \pm 0.30$$

$$\text{b) } 6.40 \pm 0.30$$

$$\text{c) } 80.0 \pm 3.2$$

$$\text{d) } 2.016 \pm 0.080$$

$$\text{e) } 320 \pm 13$$

$$\text{f) } 4.21 \pm 0.10$$

$$\text{g) } 18,700 \pm 5,900$$

$$\text{h) } 16.01 \pm 0.56$$

$$\text{i) } 1.496 \pm 0.020$$

$$\text{j) } 0.351 \pm 0.020$$

$$\text{k) } 3065 \pm 97$$

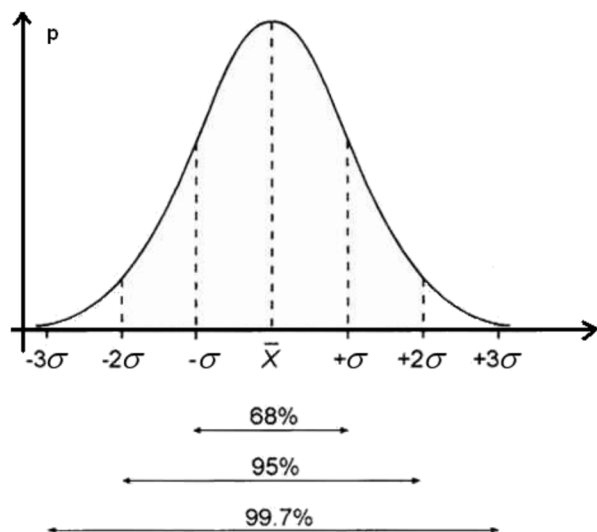
$$\text{l) } 63.62 \pm 0.90^\circ$$

$$\text{m) } 19.32 \pm 0.72$$

$$\text{n) } 23.47 \pm 0.53$$

STATISTICAL METHODS

In what was done here, it was assumed that the result of a measurement is between $x - \Delta x$ and $x + \Delta x$ and that all the values in this range are equally likely. On the other hand, there are sometimes reasons to believe that in the values in the centre of the interval are more probable than those near the edges of the interval. In this case, it is always assumed that the probability for the different values on the interval is given by this graph (called a Gaussian distribution).



This distribution is described with a standard deviation σ . This distribution means that the value has a 68% chance of being between $x - \sigma$ and $x + \sigma$, 95% chance of being between $x - 2\sigma$ and $x + 2\sigma$, and a 99.7% chance of being between $x - 3\sigma$ and $x + 3\sigma$. The uncertainty used will, therefore, depend on the level of confidence wanted. Very often, the uncertainty is equal to 2σ for a 95% confidence level.

Then, these probabilities change completely how to calculate uncertainties. So, here's what is obtained (without a proof that exceeds the level of these notes) for a function of two variables.

$$\Delta z = \sqrt{\left(\frac{\partial z}{\partial x} \Delta x\right)^2 + \left(\frac{\partial z}{\partial y} \Delta y\right)^2}$$

Here's what this formula gives for basic operations.

Addition $z = x + y$ or subtraction $z = x - y$

$$\Delta z = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Multiplication $z = xy$ or division $z = x/y$

$$\frac{\Delta z}{z} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$$

You probably recognize the uncertainty equations used in chemistry.

GRAPHICAL DATA ANALYSIS

Sometimes, the parameters of an equation will be found with a graph. For example, the formula giving the position versus time in a uniform rectilinear motion is

$$y = y_0 + vt$$

where v is the velocity and y_0 is the position at $t = 0$.

To determine the speed, the position of the object at different instants can be measured to plot a graph of the position versus time. Then, the slope of this graph is the speed of the object.

The graph

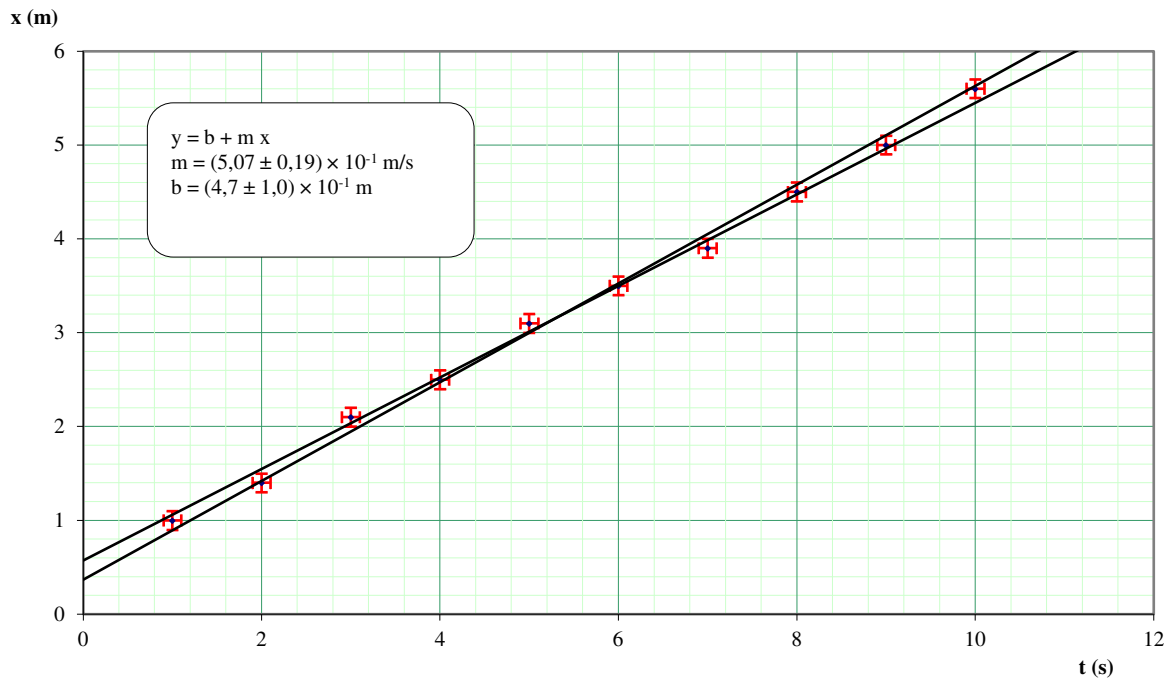
Suppose that the measurements of the position are

TABLE 1 *Position as a Function of Time*

Time (t)	Position (x)
s	m
± 0.1	± 0.1
1.0	1.0
2.0	1.4
3.0	2.1
4.0	2.5
5.0	3.1
6.0	3.5
7.0	3.9
8.0	4.5
9.0	5.0
10.0	5.6

The graph corresponding to these data is on the next page (Figure 1). We'll see later how to create such a graphic with Excel.

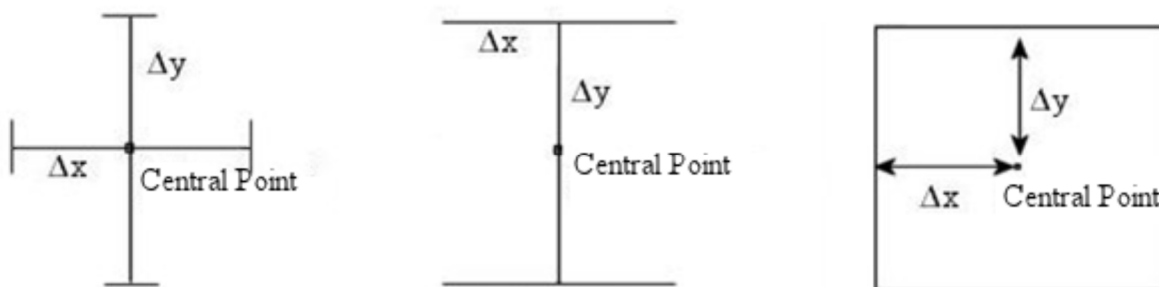
Graph 1: position as a function of time



Some explanations are needed here.

Points Representation

As the values of the points on the graph have uncertainties, these uncertainties must be shown on the graph. There are three ways to represent these uncertainties.



In the first two cases, the uncertainties are represented by lines, called uncertainty bars, whose length, from the central point, is equal to the uncertainty. The uncertainty bars are, however, arranged in different ways in the first two figures. The third way is to draw a rectangle of uncertainty whose borders are located at a distance Δx from the centre point in the x -direction and at a distance Δy from the centre point in the y -direction.

In all cases, the uncertainty bars and the rectangle of uncertainty delimit a rectangular region where the real value of the measurement can be found, taking into account the uncertainties. In the first two cases, the limits of the rectangular area are more subtle but they are identical to the rectangle of uncertainty. It takes just a little more imagination to see these limits.

Determination of the Equation of the Line

If the phenomenon is linear, then there must be a straight line which touches all the experimental points. However, since the experimental data are not simple points but rather a set of rectangles of uncertainty, several straight lines can possibly pass through all the rectangles of uncertainty. All the lines passing through all the rectangles of uncertainty are possible lines according to the experimental results. However, there is no need to plot all of these lines; only two will do. These two lines represent the two extreme cases: the line having the maximum slope and the line having the minimum slope. All the other lines have a slope whose value lies between these two limits.

Maximum slope: For the maximum slope, the line passing through all the rectangles of uncertainty having the largest possible slope is plotted

Minimum slope: For the minimum slope, the line passing through all the rectangles of uncertainty having the smallest possible slope is plotted.

These are the two lines plotted on the graph.

Excel will calculate the values of maximum and minimum slopes for you. (If you had plotted this graph by hand, you would just take two points on each of these lines to calculate the slopes.) In our example, the values are

$$\begin{aligned}\text{slope}_{\max} &= 0.5258 \frac{\text{m}}{\text{s}} \\ \text{slope}_{\min} &= 0.4883 \frac{\text{m}}{\text{s}}\end{aligned}$$

Excel, therefore, gives the following value for the slope.

$$\text{slope} = 0.507 \pm 0.019 \text{ m/s}$$

Excel has also calculated the maximum and minimum values of all possible vertical axis intercept. In our example here, these extreme values are

$$\begin{aligned}b_{\max} &= 0.57 \text{ m} \\ b_{\min} &= 0.37 \text{ m}\end{aligned}$$

Excel, therefore, gives the following value for the intercept.

$$b = 0.47 \pm 0.10 \text{ m.}$$

Thus, all the parameters of the linear equation (the slope and the intercept) with their uncertainties are known.

Plotting the Graph with Excel

With the “graphique de science” add-in, graphics can easily be plotted with all the aforementioned specifications. Moreover, the add-in will calculate the slope and the intercept values with their uncertainties.

First, the data must be entered in Excel. Write all the values of both variables in two different columns and all the values of the uncertainty of both variables in two other different columns. If the value of the uncertainty is constant, you don't need to write the values. The order of the columns is not important.

Once this is done, click on the “graphique de science” icon. Of course, the add-in must be installed on the computer for the icon to appear. Then, the following dialogue box appears.

Graphique de science

Plage pour l'axe des x

Plage pour les incertitudes des x

Plage pour l'axe des y

Plage pour les incertitudes des y

incertitude de x

☐ inc. de x, constante
☐ inc. de x, % constant
☒ inc. de x, plage

Valeur inc.

incertitude de y

☐ inc. de y, constante
☐ inc. de y, % constant
☒ inc. de y, plage

Valeur inc.

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OK Annuler

Now, you must specify where the data are located. Click in the white space immediately below “Plage pour l’axe des x” and then select the data for the horizontal axis. Then click in the white space under the indication “Plage pour les incertitudes des x” and then select the values of the uncertainties of the data for the horizontal axis. Proceed in the same manner for the values and the uncertainties of the vertical axis (y-axis).

If the uncertainty of a variable is constant, the value of uncertainty can be indicated instead of selecting a column of numbers. For example, if the value of the uncertainty of x is constant, then you can select the button “Inc. x, constant” in the box “incertitude de x”. The white area under “Plage pour les incertitudes des x” will become grey and inactive while the grey area under “Valeur Inc.” will become white and active. Then just write the value of the uncertainty in this white area.

Once the values of all variables are given, click OK. The graph will appear with the values and the uncertainties of the slope and the intercept.

What if no Line Passes through all the Rectangles of Uncertainty?

Sometimes, no line passes through all the rectangles of uncertainty. In this case, there are three options.

Maybe there is no line because one measurement is simply wrong. For example, 10.7 can have been noted instead of 1.07 for a measurement. Thus, the line cannot touch at this rectangle. This value can then be removed from your data in order to make the graph again, provided you have a very good reason to believe that an error has occurred. It would be a shame to miss out on an unexpected discovery only because you thought the value was wrong. (This happened to a lot of great scientists by the way.)

Maybe there is no line because the rectangles of uncertainty are too small. All the points seem to be located more or less along a straight line, but no line can touch all the rectangles of uncertainties because they are too small. In this case, the problem may come from an underestimation of the uncertainties. It would probably be a good idea to reassess the uncertainties to see whether they were too small.

EXERCISES

With the following data, determine the value of the slope and the intercept (with their uncertainties) of the line.

x	y
3.17 ± 0.15	11.12 ± 0.20
6.02 ± 0.20	12.31 ± 0.40
7.49 ± 0.25	12.47 ± 0.60
10.22 ± 0.30	13.56 ± 0.80
12.07 ± 0.35	14.0 ± 1.0
15.05 ± 0.40	16.5 ± 1.2

Answer Slope = 0.403 ± 0.083 intercept = 9.83 ± 0.52

STATISTICAL METHODS

It is possible to find the best straight line that goes through the points. Excel can easily find the equation of this line (by the method of least squares). Be aware that the uncertainty of the slope and the y-intercept can also be calculated. The formulas are rather heavy and depend on whether you consider that the values have no uncertainties, have an uncertainty or a statistical uncertainty (in which the values are more likely in the centre of the interval). These formulas will not be given but they can be found Éric Laflamme's excellent book *Mesures et incertitudes en laboratoire*.

ANALYSIS

1st Part: Comparing the Experimental and Theoretical Results

After obtaining the final results from calculations and graphics, you have to determine if these results agree with the theoretical values or if the results obtained by two different methods agree with each other. However, the two values to compare each has an

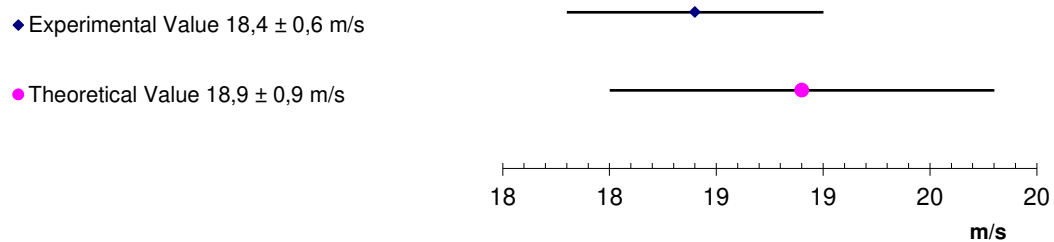
uncertainty. How is it possible to determine if they agree while taking into account the uncertainties? The rule is very simple. If the values and their uncertainties have shared values, then they agree.

Graphical Method

One of the ways to make the comparison is to use a graph to see if the values overlap. For example, suppose that the two values to compare are:

Experimental Speed: $18.4 \text{ m/s} \pm 0.6 \text{ m/s}$
 Theoretical Speed: $18.9 \text{ m/s} \pm 0.9 \text{ m/s}$

These values can be represented as follows.

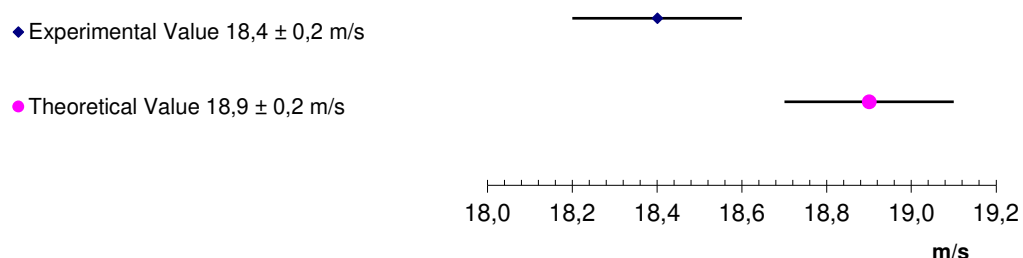


The two values have common values between 18.0 m/s and 19.0 m/s. In this case, the two values agree with each other.

Now suppose that the two values to compare are rather

Experimental Speed: $18.4 \text{ m/s} \pm 0.2 \text{ m/s}$
 Theoretical Speed: $18.9 \text{ m/s} \pm 0.2 \text{ m/s}$

These values can be represented as follows.



Here, the values disagree since the maximum value of the experimental speed (18.6 m/s) does not reach up to the minimum value of the theoretical speed (18.7 m/s).

Calculation method

With the previous examples, it can be inferred that the two results agree if the uncertainty bars meet, and they disagree if they do not meet. A simple calculation can be done to check whether the uncertainty bars are long enough for the two results to agree or not.

If the difference between the values $<$ sum of the uncertainties, then they agree

If the difference between the values $>$ sum of the uncertainties, then they disagree

Let's do a test with the previous examples. In the first case, the difference between the values is $18.9 - 18.4 = 0.5$ while the sum of the uncertainties is $0.6 + 0.9 = 1.5$. Since the difference is smaller than the sum of uncertainties, those two results agree. In the second case, the difference between the values is still $18.9 - 18.4 = 0.5$ while the sum of the uncertainties is now $0.2 + 0.2 = 0.4$. Since the difference between the values is larger than the sum of uncertainties, those two results disagree.

Results That Strongly Disagree

If two results strongly disagree (e.g., an experimental gravitational acceleration of 2 m/s^2 vs a theoretical value of 9.8 m/s^2) you should stop and think. No laboratory that will be done here is supposed to lead to results that strongly disagree. Check your calculations.

If you don't find any calculation errors, leave at least a comment showing that you understand that this result is ridiculous. If you move on after a ridiculous result without leaving any comment, your grade will be reduced.

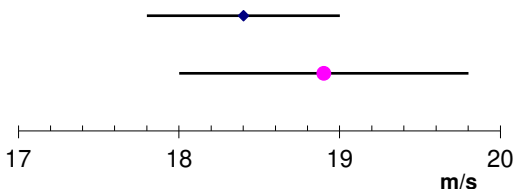
2nd Part (If the Values Agree Only): Accuracy of the Method Used

If the results agree, then a judgment can be made on the quality of this agreement by considering the accuracy of the results. Usually, this is done by comparing the uncertainty of the measurement and the measurement itself. If the uncertainty represents less than 5% of the measurement, the method can be considered relatively accurate. Of course, the desired precision depends on the context. In a research laboratory, more precision would be required.

Let's take two examples to illustrate. Let's assume first that the results are

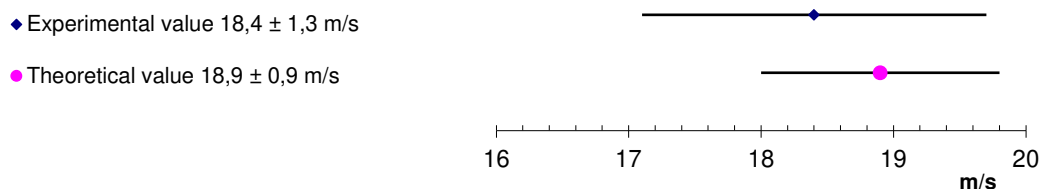
◆ Experimental Value $18,4 \pm 0,6 \text{ m/s}$

● Theoretical Value $18,9 \pm 0,9 \text{ m/s}$



The uncertainty on the experimental value is 3.2%, and the uncertainty on the theoretical value is 4.7%. As these two percentages are lower than 5%, it can then be said that the experiment is accurate.

Now, let's assume that the two results are as follows.



The uncertainty on the experimental value is 7.1%, and the uncertainty on the theoretical value is 4.7%. As one of the percentages is greater than 5%, it can be said that the experiment is inaccurate.

3rd Part: Critical Look at the Experiment

In this section, list all those things that can affect your measurements. Usually, these elements are

- 1- Factors that have been neglected (e.g., friction).
- 2- Explanation of the constraints on the application of a law (the law is an approximation).
- 3- An element of the experiment that was poorly designed.

The latter often refers to a measurement that was difficult to carry out.

If possible, the analysis should describe the qualitative effect of each of the elements (determine if this element is supposed to make the measured value larger or smaller). For example, if the gravitational acceleration of an object in free fall was measured, you will probably say that frictional forces will make your measured acceleration smaller than expected.

4th Part (only if the values disagree): Causes of the Disagreement

In this section, try to explain why the values disagree. If all the results agree, it is, of course, useless to explain the discrepancy!

Of course, you can invoke to the causes mentioned in the section “critical look” but be careful of the way these causes change the measured value. For example, in an experiment on mechanics, let's suppose that the measured value for the speed is greater than expected. Then, it would be weird to acknowledge friction as being responsible for the discrepancy because, usually, the frictional force is opposed to the motion and ought

to slow down the object instead of speeding it up.

The causes can vary greatly depending on the experiment, and it is, therefore, difficult to give guiding lines to follow to determine the cause.

Do not mention the uncertainties of the measurements. For example, do not say that the disagreement is due to the fact that the distances were measured with a rule and the rule had an uncertainty of 1 mm. Since a calculation of the uncertainty on the final result was done, this lack of precision is already taken into account. If the results disagree, they disagree even if these uncertainties on the measures were taken into account. Therefore, find other causes of disagreement.

Most of the disagreeing results come from calculation errors but you should never mention calculation errors as the cause of the disagreement. If you think that a calculation error is the cause, then check your calculations and correct the errors! If you do not find any error, then you cannot say that a calculation error is the cause since you have no reason to suspect there was an error. You will SO lose points if you say something like “there may have been a calculation error”. Your instructor will read this sentence as “I really don’t care enough about this experiment to go back and check my work, so please butcher my lab grade.”

You can say that there was a manipulation error, but it is not enough to say that there was a faulty manipulation. You have to say which error was made and why it leads to an inaccurate result. If you do not spot any error, do not talk about manipulation errors since you have no reason to think that there was an error. In fact, it is best to avoid blaming yourselves unnecessarily since it destroys your credibility. If you are reading a scientific paper in which the author tells you that he is a poor experimenter and that he surely made an error somewhere but does not know where, will you read the next paper from this author? You need to act as if you were infallible, even though we know that this is not always true. You will SO lose points if you use the words “measurement error” or “human error” vaguely. Your instructor will read this sentence as “I don’t care enough about this experiment to actually think about what is going on in it, so please butcher my lab grade.”

5th Part: Proposed Improvements

You must always suggest ways to improve the experiment. If the results disagree, just propose a method to redo the experiment while trying to correct the problem by changing the setup or by changing the theoretical formula in order to take into account other factors. Even if the results agree, you should propose ways to improve the accuracy of the experiment by finding ways to have measurements with smaller uncertainties. Of course, the value of the measurement that contributes the most to the final uncertainty should be improved first. For example, at the beginning of this introduction, there was an example in which the height of a cliff was determined by measuring the time that took an object to reach the bottom of the cliff. In this example, the uncertainty of the height came from the uncertainty of the time and the uncertainty of g . As the final uncertainty came mostly from

the uncertainty of the time, a way to improve the measurement of time should be proposed in this case. It would be almost useless to increase the accuracy of the value of g .

THE CONCLUSION OF THE LAB REPORT

The conclusion is usually very short. Just state the goals of the experiment and state whether or not the goal has been achieved. Depending on the agreement and the accuracy, the conclusion is worded differently. Here are some examples of conclusions of a report whose goal was to measure the speed of sound.

1) Results agree and are accurate

(Say theoretical speed = 330 ± 5 m/s and experimental speed = 334 ± 10 m/s)

In this experiment, the formula for the speed of sound has been verified. The speed obtained experimentally agrees with the theoretical speed, and the results are accurate. The results confirm the validity of the formula.

2) Results agree and are not accurate

(Say theoretical speed = 330 ± 5 m/s and experimental speed = 334 ± 55 m/s)

In this experiment, the formula for the speed of sound has been verified. The speed obtained experimentally agrees with the theoretical speed, but the value of the experimental speed is not very accurate. The results confirm the validity of the formula, but the accuracy should be improved to obtain a better verification.

3) Results disagree

(Say theoretical speed = 330 ± 5 m/s and experimental speed = 360 ± 15 m/s)

In this experiment, the formula for the speed of sound has been verified. The speed obtained experimentally disagrees with the theoretical speed. Therefore, the results do not confirm the validity of the formula. This discrepancy is probably due to...

Here's an example of a lab report.

LAB 0

THE PENDULUM

GOAL

Check the formula giving the period of a pendulum.

THEORY

The period of oscillation of a pendulum is given by

$$T = 2\pi \sqrt{\frac{\ell}{g}} = \frac{2\pi}{\sqrt{g}} \ell^{\frac{1}{2}}$$

Note that this period is independent of the mass of the pendulum.

METHOD USED

To verify this law, the period of oscillation of a pendulum will be measured for different lengths of rope. With the graph of the linearized equation, it will be relatively easy to see if the experimental results agree with the theoretical predictions. The period of 50 oscillations will be measured as this greatly decreases the uncertainty of the period.

EQUIPMENT

- Rope
- Masses ($\pm 1\%$)
- Stopwatch
- Ruler

PROCEDURE

- Hang a 100 g mass at the end of a rope.
- Measure the time required to make of 50 oscillations for each length of the rope on the following table.

LENGTH	TIME
cm	s
$\pm$	$\pm$
25	
30	
35	
40	
45	
50	
55	
60	
65	
70	

MEASUREMENTS

In the lab report, give the following values:

- The length and the time for 50 oscillations (present these values in a table).

CALCULATIONS

- Calculate the period (time for 1 oscillation) for each length and present the results in a table.

A linearized graph must then be plotted. As the equation for the period of a pendulum is not linear, a change of variable must be performed to obtain a straight line. Here is the change of variable.

$$T = \ell^{1/2} \frac{2\pi}{\sqrt{g}}$$

$$\ln T = \ln \ell^{1/2} + \ln \frac{2\pi}{\sqrt{g}}$$

$$\ln T = \frac{1}{2} \ln \ell + \ln \frac{2\pi}{\sqrt{g}}$$

In fact, a better version can be done to avoid problems of units.

$$T = \ell^{1/2} \frac{2\pi}{\sqrt{g}}$$

$$\frac{T}{1s} = \frac{\ell^{1/2}}{1s} \frac{2\pi}{\sqrt{g}}$$

$$\frac{T}{1s} = \frac{\ell^{1/2}}{1m^{1/2}} \frac{2\pi}{\sqrt{g}} \frac{1m^{1/2}}{1s}$$

$$\ln \left(\frac{T}{1s} \right) = \ln \left(\frac{\ell^{1/2}}{1m^{1/2}} \right) + \ln \left(\frac{2\pi}{\sqrt{g}} \frac{1m^{1/2}}{1s} \right)$$

$$\ln \left(\frac{T}{1s} \right) = \frac{1}{2} \ln \left(\frac{\ell}{1m} \right) + \ln \left(\frac{2\pi}{\sqrt{g}} \frac{1m^{1/2}}{1s} \right)$$

If $u = \ln(\ell / 1m)$ and $v = \ln(T/1s)$, then the equation becomes

$$v = \frac{1}{2} u + \ln \left(\frac{2\pi}{\sqrt{g}} \frac{1m^{1/2}}{1s} \right)$$

This is the equation of a straight line with the following characteristics.

$$\text{Slope} = 1/2 \qquad \text{Intercept} = \ln \left(\frac{2\pi}{\sqrt{g}} \frac{1m^{1/2}}{1s} \right)$$

- Calculate the values of u and v with $u = \ln(\ell / 1m)$ and $v = \ln(T/1s)$. Present the results in a table.
- Plot the graph of v as a function of u .

- Calculate the theoretical value of the intercept

$$\ln\left(\frac{2\pi}{\sqrt{g}} \frac{1m^{1/2}}{1s}\right)$$

using $g = 9.81 \text{ m/s}^2 \pm 0.01 \text{ m/s}^2$.

Analysis

- Compare the value of the theoretical slope ($1/2$) with the value of the slope on the graph.
- Compare the value of the theoretical intercept with the value of the intercept on the graph.

Verification of the Law of the Pendulum

Luc Tremblay, Isaac Newton, Leonhard Euler
Collège Mérici, Quebec City

November 17, 1722

Abstract

In this report, the formula giving the period of a pendulum will be verified. The verification is done by measuring the period of oscillation for pendulums of different lengths. With a graph, it is possible to see if the law is valid. The results obtained experimentally disagrees with the theoretical results. This seems to come from a faulty measurement of the length of the rope, from air friction acting on the pendulum, and from the fact that the formula for the period of the pendulum is obtained by making some assumptions that do not correspond to the experimental setup used.

Measurements

Time as a Function of the Length of the Rope

LENGTH	TIME FOR 50 OSCILLATIONS
m	s
± 0.002	± 0.3
0.250	49.9
0.300	54.7
0.350	59.2
0.400	62.8
0.450	67.5
0.500	70.8
0.550	74.7
0.600	78.2
0.650	81.2
0.700	84.4

Calculations and graphs

Oscillation Period

Calculation example (with data from the first line of the table)

$$\begin{aligned}T_{1 \text{ oscillation}} &= \frac{T_{50 \text{ oscillations}}}{50} \\&= \frac{49.9s}{50} \\&= 0.998s\end{aligned}$$

$$\begin{aligned}\Delta T_{1 \text{ oscillation}} &= \frac{1}{50} \Delta T_{50 \text{ oscillations}} \\&= \frac{1}{50} \cdot 0.3s \\&= 0.006s\end{aligned}$$

$$T_{1 \text{ oscillation}} = 0.998s \pm 0.006s$$

Period as a Function of the Length of the Rope

LENGTH	PERIOD
<i>m</i>	<i>s</i>
± 0.002	± 0.006
0.250	0.988
0.300	1.094
0.350	1.184
0.400	1.256
0.450	1.350
0.500	1.416
0.550	1.494
0.600	1.564
0.650	1.624
0.700	1.688

u and v calculation

u calculation example (with data from the first line of the table)

$$\begin{aligned}u &= \ln\left(\frac{\ell}{1m}\right) \\&= \ln\left(\frac{0.25m}{1m}\right) \\&= -1.3863\end{aligned}$$

$$\begin{aligned}\Delta u &= \frac{\partial u}{\partial \ell} \Delta \ell \\&= \frac{1}{\ell} \Delta \ell \\&= \frac{1}{0.25m} \cdot 0.002m \\&= 0.0080\end{aligned}$$

$$u = -1.3863 \pm 0.0080$$

v calculation example (with data from the first line of the table)

$$\begin{aligned}v &= \ln\left(\frac{T}{1s}\right) \\&= \ln\left(\frac{0.988s}{1s}\right) \\&= -0.0121\end{aligned}$$

$$\begin{aligned}\Delta v &= \frac{\partial v}{\partial T} \Delta T \\&= \frac{1}{T} \Delta T \\&= \frac{1}{0.988s} \cdot 0.006s \\&= 0.0061\end{aligned}$$

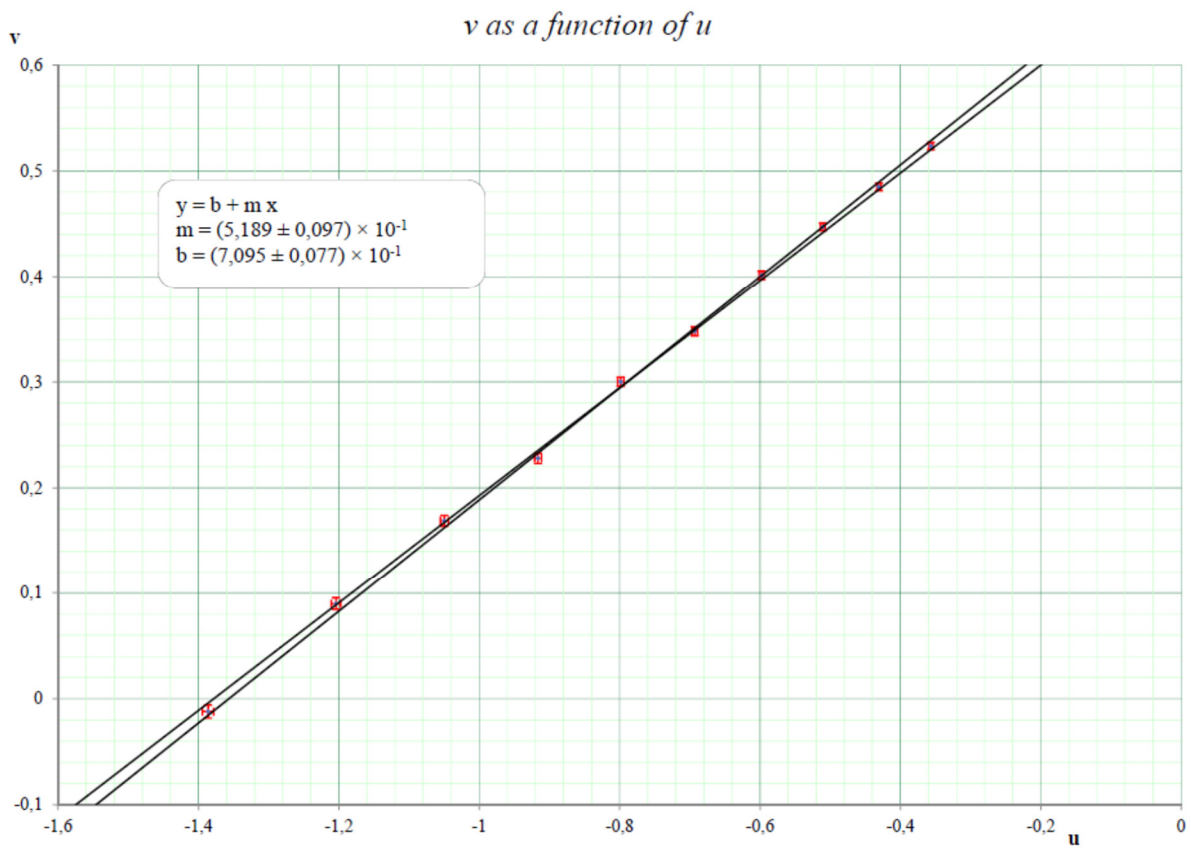
$$v = -0.0121 \pm 0.0061$$

The following results are thus obtained for u and v .

v as a function of u

<i>u</i>	<i>v</i>
-1.3863 ± 0.0080	-0.0121 ± 0.0061
-1.2040 ± 0.0067	0.0898 ± 0.0055
-1.0498 ± 0.0057	0.1689 ± 0.0051
-0.9163 ± 0.0050	0.2279 ± 0.0048
-0.7985 ± 0.0044	0.3001 ± 0.0044
-0.6931 ± 0.0040	0.3478 ± 0.0042
-0.5978 ± 0.0036	0.4015 ± 0.0040
-0.5108 ± 0.0033	0.4472 ± 0.0038
-0.4308 ± 0.0031	0.4849 ± 0.0037
-0.3567 ± 0.0029	0.5235 ± 0.0036

Here is the graph of *v* as a function of *u*.



The slope of the line on the graph is

$$\text{Slope} = 0.5189 \pm 0.0097$$

and the intercept is

$$b_{\text{exp}} = 0.7095 \pm 0.0077$$

The theoretical value of the intercept is

$$b = \ln \left(\frac{2\pi}{\sqrt{g}} \frac{1m^{1/2}}{1s} \right) = \ln \left(\frac{2\pi}{\sqrt{9.81 \frac{m}{s^2}}} \frac{1m^{1/2}}{1s} \right) = 0.69618$$

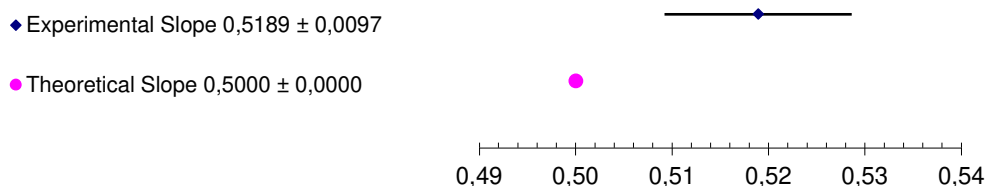
$$\Delta b = \left| \frac{\partial b}{\partial g} \right| \Delta g = \frac{\pi g^{-3/2}}{2\pi g^{-1/2}} \Delta g = \frac{\Delta g}{2g} = \frac{0.01 \frac{m}{s^2}}{2 \cdot 9.81 \frac{m}{s^2}} = 0.00051$$

$$b_{\text{theo}} = 0.69618 \pm 0.00051$$

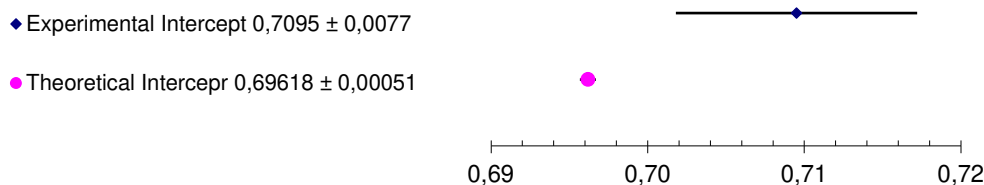
ANALYSIS

Comparison of Experimental and Theoretical Values

The theoretical slope (0.5) disagrees with the value of the experimental slope (0.5189 ± 0.0097) since the difference between the values (0.0189) is larger than the sum of the uncertainties (0.0097). The following graph shows this.



The experimental intercept (0.7095 ± 0.0077) disagrees with the theoretical intercept (0.69618 ± 0.00051) since the difference between the values (0.01332) is larger than the sum of the uncertainties (0.00821). The following graph shows this.



Critical look at the experiment

The following problems can be noted for this experiment:

- 1) The theoretical formula used in this experiment is the equation of an idealized pendulum, i.e., a point mass tied at the end of a massless string. This was obviously not the case here. The mass was not punctual, and the rope had a mass. The formula used is just an approximation for the pendulum used in this experiment. It would have been better to use a formula for the period of a pendulum that exactly matches with the pendulum used in this experiment. As we don't have the exact formula, it's hard to say how this approximation influences the value of the slope or the intercept.
- 2) The formula of the period is also an approximation that was obtained by assuming that the angular amplitude is small. In this case, the oscillation is approximately harmonic and the formula of the period is then obtained from this assumption. In the real world, the oscillations of the pendulum are not harmonic oscillations and it would have been better to find the formula that gives the exact oscillation period. The approximation is good enough when the angular amplitude is small, but it's still an approximation. As we don't have the exact formula, it's hard to say how this approximation influences the value of the slope or the intercept.
- 3) Also, air friction acts on the bob of the pendulum and its effect was not considered. The force of friction increases the period, which increases the values of T and ν . This increase should normally be greater for larger values of ν since the speed is greater for longer strings (if the angular amplitude was kept constant...). Frictional forces should, therefore, increase the slope and the y-intercept.
- 4) The mass used was not a point mass, and the actual length of the rope, which is the distance between the pivot and the centre of mass of the bob of the pendulum is not quite the length measured since the distance between the attachment point of the string and the centre of mass of the bob was not included. The values of the length (ℓ) used here are therefore too small. This decreases the values of u and the effect is greater for small values of u . As smaller values of u are more shifted down than the larger values, the slope and the y-intercept obtained will be larger than expected.

Causes of the Disagreement

A slope and a y-intercept greater than expected were obtained. As stated previously, this difference could be explained by the poor way ℓ was measured and by the friction acting on the pendulum.

It could also be explained by the fact that the formulae used are approximations but it is unclear whether this makes the slope and the intercept greater or smaller than expected.

The aforementioned effects are not really large so that the deviations from the theoretical values are small here. If these effects had been important, a straight line would not have been obtained on the graph.

Thus, these causes could explain the slight discrepancy observed.

Proposed Improvements

To improve this experiment, more attention to the measurement of the length of the string should be paid in a future experiment. It would be good to know the position of the centre of mass of the bob more accurately. The position of the centre of mass would be easier to find if a symmetrical object is used.

A denser material should also be used for the bob in order to have a more compact mass and thus to approach the idealized situation in which mass is a point mass. Using gold and osmium for the bob would improve the experiment since they are the densest elements, but the improvement would not be that great since the density of the bob used is not much lower than the density of gold and osmium. Moreover, the tremendous increase in costs that would result does not allow us to make this kind of improvement.

Finally, as the theoretical formula corresponds to a pendulum with a massless rope, the least massive rope should be used.

CONCLUSION

In this experiment, the formula for the period of a pendulum has been verified. The results obtained experimentally disagrees with the theoretical results. Therefore, the results do not confirm the validity of the formula. This seems to come from a faulty measurement of the length of the rope, from frictional forces acting on the pendulum, and from the fact that the formula for the period of the pendulum is obtained by making a few assumptions that do not correspond to our experimental setup.

Comments on the Report

There is no introduction (theory, method, equipment, and procedures) to make.

Abstract

The abstract is a short paragraph giving:

- 1- The goal of the laboratory
- 2- A very brief description of the method
- 3- The conclusion

Measurements

All the measurements made in the laboratory, and their uncertainties must be written. There must be no calculation made on these measurements.

If you have given a result sheet at the end of the lab period, the results shown in the laboratory must be identical to those listed on the lab sheet. If the results are different, all the marks of this section will be lost (often 15%).

Calculations and Graphs

All the calculations and all the uncertainty calculations must be written. If the same calculation is repeated several times, to make a table for example, simply show the calculation of the first line of the table. Then write all the results in a table immediately after the sample calculation.

Analysis

The "Analysis" section is made up of 5 parts. These are:

1. Comparison of experimental and theoretical values
2. An assessment of the accuracy of the experiment (if the results agree)
3. Critical look at the experiment
4. Reasons why there is a disagreement (if the results disagree)
5. Ways to improve the experiment.